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Positional games on graphs

THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

by

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The research work for this thesis has been carried out
under the supervision of Prof. Noga Alon and Prof. Michael Krivelevich

Submitted to the senate of Tel Aviv University
July 2007

Acknowledgements

First and foremost I wish to express my deepest gratitude to my advisor Prof. Michael Krivelevich for his guidance and help during the work on this thesis; for always having the time to see me; for looking pleased with my work even when I wasn't; for finding the perfect balance between pushing too hard and not pushing at all.

I would also like to thank my co-advisor Prof. Noga Alon for all his help.

I would like to thank my coauthors, Miloš Stojaković and Tibor Szabó, for their many contributions to this work. I have greatly benefited from their knowledge.

I would like to thank Miloš Stojaković and Eyal Lubetzky for giving me some LaTex tips and to Sonny Ben-Shimon for helping me with WORD. They helped making this thesis fancier.

I wish to thank my fellow graduate students Vera Asodi, Sonny Ben-Shimon, Eyal Lubetzky, Liam Roditty, Asaf Shapira, Uri Stav, Oded Schwartz and Danny Vilenchik for many valuable discussions and for creating a lively working environment.

I wish to thank my family for their love and support.

Abstract

The theory of Positional Games is an exciting and relatively young area of Combinatorics, though its origins can be traced back to Classical Game Theory. Its goal is to give a mathematical framework for analyzing games of thought. Positional games are strongly related to several other branches of Combinatorics such as Ramsey Theory, Extremal Graph and Set Theory and the Probabilistic Method. The subject has proven to be quite instrumental in deriving important results in Theoretical Computer Science, in particular in derandomization and algorithmization of important probabilistic tools.

Although sporadic papers on positional games appeared much earlier, the beginning of their systematic study can be attributed to two, by now classical, papers: that of Hales and Jewett from 1963 [41], and that of Erdős and Selfridge from 1973 [36]. The significance of these two papers certainly cannot be overestimated and goes far beyond the realm of Positional Games: the Hales-Jewett theorem is one of the cornerstones of modern Ramsey Theory, the Erdős-Selfridge argument was essentially the first derandomization procedure, a central concept in the theory of algorithms.

This thesis contains several novel contributions to the theory of Positional Games such as new instances of the Erdős paradigm, a general criterion for Avoider's win in Avoider-Enforcer games, surprising results regarding the monotonicity of such games, a study of the duration of play, game theoretic strengthening of results in Random Graphs Theory, a new class of hybrid games that we call *Bart-Moe* games and more. This thesis also includes a result which is not directly connected to games; it is a novel sufficient condition for Hamiltonicity of relatively sparse graphs, based on expansion and high connectivity. It is included because its proof was found through research in Positional Game Theory, and later on, this criterion had found many other applications, in Positional Game Theory as well as in other branches of Combinatorics.

Contents

Chapter 1

Introduction

The theory of Positional Games is a branch of Combinatorics, whose main aim is to develop systematically an extensive mathematical framework for a variety of two player perfect information games, ranging from such commonly popular games as Tic-Tac-Toe and Hex to purely abstract games played on graphs and hypergraphs. Though a "close relative" of the classical Game Theory of von Neumann and of Nim-like games popularized by Conway and others (see the excellent book [24]), Positional Games are quite different and are more combinatorial in nature. Positional games are strongly related to several other branches of Combinatorics such as Ramsey Theory, Extremal Graph and Set Theory and the Probabilistic Method. The subject has proven to be quite instrumental in deriving important results in Theoretical Computer Science, in particular in derandomization and algorithmization of important probabilistic tools.

Although sporadic papers on positional games appeared much earlier, the beginning of their systematic study can be attributed to two, by now classical, papers: that of Hales and Jewett from 1963 [41], and that of Erdős and Selfridge from 1973 [36]. Hales and Jewett proved (in particular) that in the multi-dimensional version of Tic-Tac-Toe, played on the d-dimensional cube $[n]^d$, the first player has a (very non-explicit) winning strategy if the dimension d is large enough compared to the cube size n. Erdős and Selfridge provided a very useful general criterion for the existence of a drawing strategy for the second player in unbiased games (due to the so called Strategy Stealing argument, this yields a drawing strategy for the first player as well). The significance of these two papers certainly cannot be overestimated and goes far beyond the realm of Positional Games: the Hales-Jewett theorem is one of the cornerstones of modern Ramsey Theory, the Erdős-Selfridge argument was essentially the first derandomization procedure, a central concept in the theory of algorithms.

József Beck has turned the field of Positional Games into a well established mathematical discipline through his long and impressive series of papers on the subject, spanning about a quarter of a century. Beck was the first to analyze systematically biased games, to provide a criterion for Maker's win in Maker-Breaker games, to find a game theoretic version of the second moment method and so on. Through his research in Positional Games he was able to derive an algorithmic version of the so called Lovász Local Lemma – undoubtedly one of the main algorithmic achievements of the Probabilistic Method. This result of Beck serves as yet another very convincing illustration of the major impact the study of Positional Games has had on Combinatorics and Theoretical Computer Science.

Positional games are finite perfect information games with no chance moves and as such can in principle be solved completely by an all-powerful computer without the aid of randomness. In reality however, this is not the case, due to the prohibitive complexity of the exhaustive search approach; this only stresses the need for accessible mathematical criteria for analyzing such games. Taking into consideration the aforementioned (theoretical) power of computers, it is very surprising that the unexpected presence of probabilistic considerations and arguments in the analysis of positional games is almost ubiquitous. This phenomenon was first indicated by Paul Erdős and is therefore referred to as the Erdős paradigm. This paradigm has since been discussed in details and masterfully implemented by József Beck.

Formally, a *positional game* is a hypergraph $(X, \mathcal{H})$, where the set X is called the "board" of the game and $\mathcal{H} \subseteq 2^X$ is a family of *target subsets* of X. During the game, two players take turns selecting previously unclaimed elements of the board; each player claims one element of the board per round. There are several types of positional games with different rules for determining the winner.

In a *strong* positional game, $\mathcal{H}$ is referred to as the family of "winning sets". The first player to claim an element of $\mathcal{H}$ is the winner (if this does not happen, then the game ends in a draw). This is the most natural type of games, but is also the hardest to analyze and very little is known about it. Results from Ramsey Theory and the notion of strategy stealing (an existence argument from classical Game Theory) are of utmost value here. A classical though trivial example of a strong game is Tic-Tac-Toe (Noughts and Crosses). Another famous example is 5-in-a-row on an unbounded board.

In a *Maker/Breaker-type* positional game, the two players are called Maker and Breaker and $\mathcal{H}$ is referred to as the family of "winning sets". Maker wins the game if the set he has claimed by the end of the game (that is, when every element of the board has been claimed by one of the players) contains a winning set, that is, an element of $\mathcal{H}$; otherwise Breaker wins. These games can be considered as a relaxation of strong games (and are therefore sometimes referred to as *weak games*), but are also interesting in their own right - a classical example of the Maker-Breaker setting is the popular board-game Hex. Putting aside a few scattered results, the theory of Maker-Breaker games started with a general criterion of Erdős and Selfridge [36] for Breaker's win. Subsequently, Beck started a systematic study of Maker-Breaker games with a bias. In particular, in [6] he proved a generalization of the Erdős-Selfridge criterion.

In an *Avoider/Enforcer-type* positional game, the players are called Avoider and Enforcer and $\mathcal{H}$ is called the family of "losing sets". Avoider wins the game if the set he has claimed by the end of the game does not contain a losing set, that is, an element of $\mathcal{H}$; otherwise Enforcer wins. These games are considered less natural than their Maker-Breaker analog, as Avoider's goal is to *not* claim a "target" set. However, they prove to be very interesting in their own right as well as useful for analyzing Maker-Breaker games.

This thesis is comprised of four parts. In Part I we study biased games (that is games in which one of the players is allowed to claim more than one element of the board per turn) on complete graphs. The main question we are concerned with is "How large does the bias of this player has to be in order to guarantee his win?". In Part II we study (mostly) unbiased games (both players claim exactly one board element per turn) on more general families of graphs. Here we are interested, not only in the identity of the winner, but also in the duration of play and in games on graphs with certain properties. In Part III we introduce a new model of positional games, which we call Bart-Moe games, that generalizes both Maker-Breaker games and Avoider-Enforcer games. In Part IV we prove a novel criterion for Hamiltonicity. This is rather a result in Graph Theory and not in Positional Game Theory. However, it was used to derive interesting results in Positional Game Theory (some of them appearing in this thesis). Moreover, one of these results was the original motivation for obtaining this criterion. Once again this shows the impact that the study of positional games has on other branches of Combinatorics.

We now give a short description of the contents of every chapter of this thesis. We list here only the central results of every chapter, and those that are relatively easy to state without introducing too much notation and definitions. In some cases, this might make our statements less accurate than the ones mentioned in the chapters themselves.

In **Chapter 2** we study biased Maker-Breaker games. The study of Maker-Breaker games on the edges of a (complete) graph was initiated by Lehman [60] who, in particular, proved that in the game on the edge set of K_n, Maker can easily build a spanning tree (by "easily" we mean that he can do so within $n-1$ moves). Chvátal and Erdős [30] suggested to "even out the odds" by giving Breaker more power, that is, by increasing his bias. In a *biased* (p,q) Maker-Breaker game, Maker claims exactly p elements of the board per turn (instead of 1) and Breaker claims exactly q elements of the board per turn (instead of 1). It is relatively clear that having a larger bias cannot harm a player. Chvátal and Erdős have proved that the $(1,b)$ game $\mathcal{T}_n$ of "Connectivity", where the family $\mathcal{T}_n$ of winning sets consists of the edge-sets of all spanning connected subgraphs of K_n, is won by Maker even when the bias b of Breaker is as large as $(1/4-o(1))n/\log n$, whereas Breaker wins this game if his bias is at least $(1+o(1))n/\log n$. Beck [6] improved their constant $1/4$ to $\log 2$ which is currently still the best known bound. His proof relies on a generalization of the Erdős-Selfridge Theorem to biased games and a "building via blocking" trick. Chvátal and Erdős also showed that the $(1,1)$ game $\mathcal{H}_n$ of "Hamiltonicity", where Maker's goal is to build a Hamiltonian cycle (the family $\mathcal{H}_n$ of winning sets consists of the edge-sets of all Hamiltonian spanning subgraphs of K_n), is won by Maker for sufficiently large n. They also conjectured that in fact Maker can win the $(1,b)$ Hamiltonicity game for some b that tends to infinity with n. This was proved by Bollobás and Papaioannou [27], who showed that Maker wins Hamiltonicity against a bias of $O(\log n/\log\log n)$. Beck [9] gave a winning strategy for Maker against a bias of $\left(\frac{\log 2}{27}-o(1)\right)\frac{n}{\log n}$, thus determining the correct order of magnitude of the "turning point" from Maker's win to Breaker's win. Very recently, Krivelevich and Szabó [58] were able to improve Beck's constant $\frac{\log 2}{27}$ to $\log 2$ which is currently the best known bound.

Following [42] it will be convenient to introduce some notation. For a family $\mathcal{H}$ of winning sets, let $b_{\mathcal{H}}$ be the non-negative integer for which Breaker has a winning strategy in the $(1, b)$ game $\mathcal{H}$ if and only if $b \geq b_{\mathcal{H}}$. Note that $b_{\mathcal{H}}$ is well-defined for any (monotone increasing) family $\mathcal{H}$ (unless $\mathcal{H}$ contains a hyperedge of size at most one). We call $b_{\mathcal{H}}$ the *threshold bias* of the game $\mathcal{H}$. By the above, $(\log 2 - o(1))n/\log n \leq b_{\mathcal{H}_n} \leq b_{\mathcal{T}_n} \leq (1 + o(1))n/\log n$.

There is an intriguing relation between the threshold biases of the Connectivity and Hamiltonicity games and the threshold probability at which the random graph $G(n, p)$ first possesses these properties. The number of edges of the random graph $G(n, p)$ around this threshold $p = \log n/n$ has the same order of magnitude as the number of edges Maker has in his graph at the end of the $(1, b_{\mathcal{T}_n}, \mathcal{T}_n)$ game or the $(1, b_{\mathcal{H}_n}, \mathcal{H}_n)$ game. This phenomenon (known as the Erdős paradigm) was pointed out by Beck [9], where this observation is attributed to Erdős (indeed the first result, suggesting such a relation, appeared already in [30]). Since then, several other games have been found to have a threshold bias which is closely linked to a meaningful random graph threshold related to that particular game (see, e.g., [11, 16, 20, 21, 70]). Note that the asymptotic values of the threshold biases $b_{\mathcal{T}_n}$ and $b_{\mathcal{H}_n}$ are not known in general. It can still turn out that these threshold biases are asymptotically equal to the inverse of the corresponding sharp thresholds for the appropriate properties of random graphs.

We describe three new instances of this phenomenon; the "non-planarity" game $\mathcal{NP}_n$, the "non-k-colorability" game $\mathcal{NC}_n^k$, and the "K_t-minor" game $\mathcal{M}_n^t$. All three are natural graph games that have not been studied before. We prove that all three threshold biases $b_{\mathcal{NP}_n}, b_{\mathcal{NC}_n^k}$ and $b_{\mathcal{M}_n^t}$ are around $n/2$ (assuming certain bounds on the parameters k, t and n). The proofs involve an intriguing role interchange between the players.

References: The results of this Chapter were accepted for publication as:

- D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Planarity, colorability and minor games, SIAM Journal on Discrete Math., to appear [45].

In **Chapter 3** we study biased Avoider-Enforcer games. At first sight, the definition of Avoider-Enforcer games seems less natural than that of Maker-Breaker games and accordingly the theory is much less developed. We, however, argue that besides being interesting in their own right, they are essential in studying Maker-Breaker games (several examples appear in Chapter 2). Avoider-Enforcer games arise naturally whenever one would like to play (what looks like) Maker-Breaker games on a *monotone decreasing family.* For example, the set $\mathcal{P}_n$ of planar graphs on n vertices is a monotone decreasing family. The goal of "Maker" in this game is to keep his graph planar to the end of the game; hence, he can be thought of as Avoider, playing an Avoider-Enforcer game on the (monotone increasing) family of losing sets $2^{E(K_n)} \setminus \mathcal{P}_n = \mathcal{NP}_n$. Moreover, for certain Maker-Breaker games on monotone increasing families, the best known Maker strategies involve building a pseudo-random graph with certain parameters (see [38, 43]). It is proved that the particular pseudo-random properties of Maker's graph imply the graph-theoretic properties in question, entailing his win. The pseudo-randomness of a graph involves bounds on the number of edges between pairs of its vertex sets from *below* and from *above.* Hence, in such a game the family $\mathcal{H}$ is the intersection of a monotone increasing family and a monotone decreasing family (we will deal with such games further in Chapter 7).

Avoider-Enforcer games were studied in [42, 61, 62, 63].

Similarly to Maker-Breaker games, one would like to define for each monotone increasing family $\mathcal{H}$ the Avoider-Enforcer threshold bias $f_{\mathcal{H}}$. A reasonable choice for $f_{\mathcal{H}}$ would be the non-negative integer for which Avoider wins the $(1, b)$ game $\mathcal{H}$ if and only if $b \geq f_{\mathcal{H}}$. While the similar threshold $b_{\mathcal{H}}$ does exist for Maker-Breaker games on any hypergraph, for Avoider-Enforcer games it generally does not (see [42]). We will discuss this surprising result in Chapter 3.

Following [42], it will be convenient to introduce some (more) notation. For a hypergraph $\mathcal{H}$ we define the *lower threshold bias* $f_{\mathcal{H}}^-$ to be the largest integer such that Enforcer can win the $(1, b)$ game $\mathcal{H}$ for every $b \leq f_{\mathcal{H}}^-$, and the *upper threshold bias* $f_{\mathcal{H}}^+$ to be the smallest non-negative integer such that Avoider can win the $(1, b)$ game $\mathcal{H}$ for every $b > f_{\mathcal{H}}^+$. Except for certain degenerate cases, $f_{\mathcal{H}}^-$ and $f_{\mathcal{H}}^+$ always exist and satisfy $f_{\mathcal{H}}^- \leq f_{\mathcal{H}}^+$. Whenever $f_{\mathcal{H}}^- = f_{\mathcal{H}}^+$, the threshold bias $f_{\mathcal{H}}$ of the Avoider-Enforcer game $\mathcal{H}$ exists and $f_{\mathcal{H}} = f_{\mathcal{H}}^+$.

We find a general sufficient condition for Avoider's win which is analogous to Beck's generalization of the Erdős-Selfridge Theorem, and prove that it is "not far" from being best possible if Enforcer's bias is "small". We use this criterion, and other interesting tools, to derive bounds on f^- and f^+ for several natural games. In particular, we prove a biased Avoider-Enforcer analog of the celebrated theorem of Lehman [60]. We use this Theorem to prove that the Avoider-Enforcer connectivity game on K_n has a threshold bias (this is currently the only natural example of an Avoider-Enforcer game that has a threshold bias) and find its exact value. Furthermore, we use it to improve a result of Lu [64], regarding forcing an opponent to pack many pairwise edge disjoint spanning trees in his graph.

We provide Enforcer with a winning strategy for the $(1, q)$ Hamiltonicity game for a large q, thus "almost" solving an open question of Beck [17]. This question was completely solved later by Krivelevich and Szabó [58]. Both results use a novel criterion for Hamiltonicity (see Theorem 8.1 in Chapter 8 of this thesis and also [44]). We find bounds on the lower and upper threshold bias of the Avoider-Enforcer "non-planarity" game $\mathcal{NP}_n$, the Avoider-Enforcer "non-k-colorability" game $\mathcal{N}C_n^k$, and the Avoider-Enforcer "K_t-minor" game $\mathcal{M}_n^t$. Note that planarity, colorability and being minor-free are monotone decreasing properties and so one can argue that these Avoider-Enforcer games are more natural than their Maker-Breaker analogs. In the proofs, we explore further the interchange of roles between Avoider, Enforcer, Maker and Breaker.

Finally, we discuss the non-bias-monotonicity of Avoider-Enforcer games as opposed to the bias-monotonicity of Maker-Breaker games. We then introduce new rules for Avoider-Enforcer games that force them to be monotone and, as a result, to have a threshold bias. Note that changing the rules of Maker-Breaker games in a similar fashion, does not affect them. We discuss the difference between Avoider-Enforcer games in the "old rules" and Avoider-Enforcer games in the "new rules". We then determine precisely the order of magnitude of the threshold bias of many interesting "new rules" Avoider-Enforcer games (complemented by results of [58], this includes the k-connectivity and Hamiltonicity games).

References: Some of the results of this Chapter are a work in progress, joint with M. Krivelevich, M. Stojaković and T. Szabó [48]. Most of the results appeared as:

- D. Hefetz, M. Krivelevich and T. Szabó, Avoider-Enforcer games, Journal of Combinatorial Theory, Ser. A. 114 (2007) 840-853 [42].
- D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Planarity, colorability and minor games, SIAM Journal on Discrete Math., to appear [45].

In **Chapter 4** we are interested in the duration of play. Certain games can be shown to be an easy win for some player. For such games, a more interesting question would be "How fast can this player win?". A simple example is the Maker-Breaker connectivity game on K_n which is won by Maker in $n-1$ moves. An even simpler example is the non-planarity game which is won by Maker within at most $3n-5$ moves, irregardless of his strategy. A much more interesting (and much harder) example is the game of Hex on a board of size $n \times n$ – it is known (via strategy stealing) that Maker wins this game, but it is not even known if Maker can win within, say, $n^2/10$ rounds or if Breaker can postpone his inevitable loss for at least, say, $10n$ rounds.

We make the following useful definitions. For a $(1,1,\mathcal{H})$ Maker-Breaker game, let $\tau_M(\mathcal{H})$ be the smallest integer t such that Maker can win the game within t moves (if the game is a Breaker's win, then set $\tau_M(\mathcal{H}) = \infty$). Similarly, for a $(1,1,\mathcal{H})$ Avoider-Enforcer game, let $\tau_E(\mathcal{H})$ be the smallest integer t such that Enforcer can win the game within t rounds (if the game is an Avoider's win, then set $\tau_E(\mathcal{H}) = \infty$).

Many known results can be stated using this notation. For example, as an immediate consequence of the result of Lehman [60], Maker has a fast winning strategy in the connectivity game. That is, $\tau_M(\mathcal{T}_n) = n-1$ for every $n \geq 4$, where $\mathcal{T}_n$ is the hypergraph whose hyperedges are the (edge sets of the) spanning trees of K_n. In Chapter 4 we obtain a similar result for the Maker-Breaker k-connectivity game, for every positive integer k: we prove that $n+1 \leq \tau_M(\mathcal{V}_n^2) \leq n+2$, and that $\tau_M(\mathcal{V}_n^k) = kn/2 + o(n)$ for every fixed $k \geq 3$, where, for every $k \geq 2$, $\mathcal{V}_n^k$ is the family of k-vertex connected spanning subgraphs of K_n. In [30], Chvátal and Erdős provide Maker with a fast winning strategy for the $(1,1,\mathcal{H}_n)$ Hamilton cycle game, showing that $\tau_M(\mathcal{H}_n) \leq 2n$, where $\mathcal{H}_n$ is the hypergraph whose hyperedges are the Hamilton cycles of K_n. In Chapter 4, we almost completely close the gap between this upper bound and the trivial lower bound of $n+1$ by showing that $\tau_M(\mathcal{H}_n) \leq n+2$. We also obtain results of this type in the Avoider-Enforcer setting. We prove that the aforementioned trivial upper bound of $3n-5$ on the duration of the planarity game is essentially tight, by showing that $\tau_E(\mathcal{NP}_n) = 3n - o(n)$. We also prove that $\tau_E(\mathcal{D}_n)$ and $\tau_E(\mathcal{T}_n)$ are "very close", where $\mathcal{D}_n$ is the family of all spanning subgraphs of K_n of positive minimum degree and $\mathcal{T}_n$ is the family of all connected spanning subgraphs of K_n; and conjecture that they are in fact equal. If true, then this could be viewed as a game theoretic analog of a well known hitting time result from Random Graphs Theory.

References: The results of this Chapter were submitted for publication as:

- D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Fast winning strategies in positional games [46].

In **Chapter 5** we consider an alternative way of "helping Breaker". Following [70] we give Breaker more power, not by increasing his bias (as in Part I), but by "thinning out" the board before the game starts. Formally, let $(X, \mathcal{H})$ be a hypergraph and let $0 \leq p \leq 1$ be a real number. We define $(X_p, \mathcal{H}_p)$ to be the hypergraph whose set of vertices X_p is obtained from X by removing every vertex of X with probability $1-p$, independently for each vertex, and whose set of hyperedges is $\mathcal{H}_p = \{A \in \mathcal{H} \ : A \subseteq X_p\}$. Note that $(X_p, \mathcal{H}_p)$ is actually a probability space of hypergraphs. Looking at the unbiased $(X_p, \mathcal{H}_p)$ game, we can discuss the probability that Maker (Breaker) wins the game.

When $p = 0$, the game is played on the empty board. When $p = 1$ we get the "original" game $(X, \mathcal{H})$ (so this is a generalization). In every other case $0 < p < 1$, the best one can hope for is that a player will almost surely (a.s. for brevity) win the game. The *threshold probability* $p_{\mathcal{F}_n}$ for the family of games $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is defined to be the probability for which an almost sure Breaker's win turns into an almost sure Maker's win, that is,

$$Pr[(X_p, (\mathcal{F}_n)_p, 1, 1) \text{ is a Breaker's win}] \to 1 \text{ for } p = o(p_{\mathcal{F}_n}),$$

and

$$Pr[(X_p, (\mathcal{F}_n)_p, 1, 1) \text{ is a Maker's win}] \to 1 \text{ for } p = \omega(p_{\mathcal{F}_n}),$$

when $n \to \infty$. Such a threshold $p_{\mathcal{F}_n}$ exists, as being a Maker's win is a monotone increasing property (see [28]).

In [70] the threshold probability for the connectivity game and the perfect matching game was determined. Moreover, it was proved that the threshold probability for the Hamiltonicity game satisfies $\frac{\log n}{n} \leq p_{\mathcal{H}_n} \leq \frac{\log n}{\sqrt{n}}$, with the conjecture that $p_{\mathcal{H}_n} = \Theta\left(\frac{\log n}{n}\right)$. This was verified in [69]. Here we strengthen this result and show that the property of Maker winning the Hamiltonicity game has a *sharp* threshold at $(1 + o(1)) \log n/n$. Compared with typical results from the theory of positional games, this result is extremely tight (we obtain good bounds on the error term). In [70], relatively tight bounds for the threshold probability of the fixed clique game (Maker wins iff he is able to claim all edges of some k-clique of $G(n, p)$, where k is a predetermined constant and n is sufficiently large) were found. In Chapter 5 we generalize their result from cliques to a large class of fixed graphs. This result is analogous to a (more general) result of Bednarska and Łuczak [20].

Note that for the Hamiltonicity game, the threshold probability for winning the game is very close to the threshold probability for the appearance of a Hamilton cycle. Hence, our result can be viewed as a *game theoretic strengthening* of the famous results concerning the Hamiltonicity of $G(n, p)$ (see e.g. [67, 52, 53]). On the other hand, for the fixed graph game, the threshold probabilities for Maker's win and for the appearance of such a graph in $G(n, p)$ may differ.

References: The results of this Chapter were submitted for publication as:

- D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Unbiased games on a random graph [47].

In **Chapter 6**, we are dealing with the question of finding the "extremal sparseness" of a graph which still guarantees Maker's win in certain games; the definition of "sparseness" may vary for different games. For example, by Lehman's Theorem we know that, for every integer $n \geq 4$, **there exists** a graph on n vertices with $2n-3$ edges ($2n-2$ if Breaker is the first player) on which Maker can win the $(1,1)$ connectivity game. On the other hand, Breaker can win this game on **every** graph on n vertices and at most $2n-4$ edges ($2n-3$ if Breaker is the first player). Clearly we cannot expect Maker to win the connectivity game on **every** graph with a "large" average degree as such a graph might be disconnected. If we change the definition of sparseness we consider, and take the board to be a highly connected graph, then by Lehman's Theorem, and the well known theorems of Tutte [75] and Nash-Williams [65], it follows that Maker (as first or second player) can win the $(1,1)$ connectivity game on every 4-connected graph, whereas there exists a 3-connected graph on which Breaker wins. For every positive integer q, we find the smallest number of edges a graph G should have for Maker to win the $(1,q)$ connectivity game, played on the edges of G.

An immediate corollary of our result concerning the Hamilton cycle game on the random graph $G(n,p)$ (see Chapter 5 of this thesis) is that there exists a graph G on n vertices and with average degree $(1+o(1))\log n$, on which Maker can win the Hamilton cycle game. In Chapter 6, we will prove that in fact a constant average degree is enough to ensure Maker's win in the Hamiltonicity game. In the proof of this theorem, we use our Hamiltonicity result from Chapter 4.

As already mentioned, we also consider other kinds of "sparseness". Thus, for example, we consider a game in which the board is an (n,d,λ)-graph and Maker's goal is to build an expander, and a game in which the board is an r-chromatic graph, and Maker's goal is to build a graph with a "high" (as a function of r) chromatic number.

References: The results of this Chapter are based on work in progress which is joint with M. Krivelevich, M. Stojaković and T. Szabó [49].

In **Chapter 7** we introduce a new type of positional games which generalizes both Maker-Breaker games and Avoider-Enforcer games.

In Maker-Breaker games, the family of winning sets is monotone increasing; in Avoider-Enforcer games, the family of winning sets is monotone decreasing (equivalently, the family of "losing sets" is monotone increasing). Frieze et al. [38] studied positional games for which the family of winning sets is the intersection of a monotone increasing family and a monotone decreasing family. Here we generalize their results to biased games, that is, when one player claims exactly p elements of the board per move instead of 1. One of the major motivating ideas behind this approach is to try and create pseudo-random graphs of the appropriate edge-density. These graphs can then be used to prove that numerous other natural games of the Maker/Breaker-type can be won by Maker.

Our setting is the following. Let $\mathcal{A}$ and $\mathcal{B}$ be hypergraphs with a common vertex set V. In a $(p, q, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game (consult the Simpsons series for the origin of the names; a more mathematical explanation is given later) the players take turns selecting previously unclaimed vertices of V. The first player, called Bart (to denote his role as Breaker and Avoider together), selects exactly p vertices per move and the second player, called Moe (to denote his role as Maker or Enforcer), selects exactly q vertices per move. The game ends when every element of V has been claimed by one of the players. Bart wins the game iff he has at least one vertex in every hyperedge $B \in \mathcal{B}$ and no complete hyperedge $A \in \mathcal{A}$.

Note that when $\mathcal{A} = \emptyset$, the game is reduced to the Maker-Breaker game $\mathcal{B}$. Similarly, when $\mathcal{B} = \emptyset$, the game is reduced to the Avoider-Enforcer game $\mathcal{A}$. Moreover, when $p = 1$ and $\mathcal{A} = \mathcal{B}$, we get the hypergraph 2-coloring game studied in [38].

We prove a sufficient condition for Bart to win the $(p, 1)$ Bart-Moe game. This criterion generalizes the ones known for Maker-Breaker, Avoider-Enforcer and 2-coloring games (for $q = 1$). We use this condition to obtain a biased version of many of the results from [38]; these include building a pseudo-random graph (according to two different definitions of pseudo-randomness) of edge density $\frac{p}{p+1}$, and then using properties of pseudo-random graphs to infer Maker's win in a variety of Maker-Breaker games. Moreover, we use our criterion to obtain some biased discrepancy results (see e.g. [72, 2]). A more general discrepancy result was recently obtained by Beck [17].

References: The results of this Chapter appeared as:

- D. Hefetz, M. Krivelevich and T. Szabó, Bart-Moe games, JumbleG and Discrepancy, European Journal of Combinatorics, 28(4) (2007) 1131–1143 [43].

Chapter 8 is the only chapter that does not deal with positional games directly, but rather with a result in Extremal Graph Theory. The reason for its inclusion in this thesis is to exemplify the impact that research in Positional Game Theory has on other branches of Combinatorics.

In this chapter we prove a sufficient condition for the existence of a Hamilton cycle, which is applicable to a wide variety of graphs, including relatively sparse graphs. In contrast to previous criteria, ours is based on only two properties; one requiring the expansion of "small" sets, the other ensuring the existence of an edge between any two disjoint "large" sets. We also discuss applications in Positional Games Theory (the original motivation for obtaining this criterion), in Random Graphs Theory and in Extremal Graph Theory.

Hamiltonicity is one of the most central notions in Graph Theory, and many efforts have been devoted to obtain sufficient conditions for the existence of a Hamilton cycle (a "nice" necessary and sufficient condition should not be expected however, as deciding whether a given graph admits a Hamilton cycle is known to be NP-complete). In this chapter we will mostly concern ourselves with establishing a sufficient condition for Hamiltonicity, which is applicable to a wide class of sparse graphs.

During the last few years, several such sufficient conditions were found (see e.g. [37, 56]). These are quite complicated at times as they rely on many properties of pseudo-random graphs. Furthermore, one can argue that these conditions are not the most natural, as Hamiltonicity is a monotone increasing property, whereas pseudo-randomness is not. The main result of Chapter 8 is a natural and simple (at least on the qualitative level) sufficient condition based on expansion and high connectivity.

Other than finding a winning strategy for Enforcer in the biased Avoider-Enforcer Hamilton cycle game – the original motivation for proving this sufficient condition for Hamiltonicity, this criterion has many other applications in Positional Game Theory (such as finding the best known bounds on the threshold bias of the Maker-Breaker Hamilton cycle game on K_n [58] and finding the threshold probability for the Hamilton cycle game on $G(n, p)$ see Chapter 5 of this thesis and also [47]) as well as in other branches of Combinatorics. This includes a new proof of the classic result of Komlós and Szemerédi, establishing the exact threshold probability for the Hamiltonicity of the random graph $G(n, p)$ [53], "nearly solving" a problem of Brandt et al [29] regarding the Hamiltonicity of f-connected graphs, and more.

References: The results of this Chapter were submitted for publication as:

- D. Hefetz, M. Krivelevich and T. Szabó, Hamilton cycles in highly connected and expanding graphs [44].

In order to keep it as self contained as possible, each chapter includes its own introduction. Moreover, whenever some external result is used, we either state it explicitly (so a theorem might have multiple appearances with a different numbering), or just refer the reader to a relevant external source. Whenever we use a result which is included in this thesis, we refer the reader to its first appearance in the thesis, as well as to a relevant external source.

Part I

Biased games

Chapter 2

Maker-Breaker games

2.1 Introduction

Let m and b be two positive integers. We are given a set X and a hypergraph $\mathcal{F} \subseteq 2^X$. During the (m, b) positional game $\mathcal{F}$, two players take turns claiming previously unclaimed elements of X. In every round, the first player claims m elements, and then the second player claims b elements. The set X is called the "board"; m and b are the biases of the first and second players respectively. For the purposes of this chapter, $\mathcal{F}$ is assumed to be monotone increasing. In a *Maker/Breaker-type* positional game, the two players are called Maker and Breaker and $\mathcal{F}$ is referred to as the family of "winning sets". Maker wins the game if the set M he has claimed by the end of the game (that is, when every element of the board has been claimed by one of the players) is a winning set, that is $M \in \mathcal{F}$; otherwise Breaker wins. Observe that, since $\mathcal{F}$ is assumed to be monotone increasing, the game could essentially be stopped as soon as Maker occupied a *minimal* winning set $F \in \mathcal{F}$; the position on $X \setminus F$ can no longer influence the outcome of the game. Hence, Breaker wins if and only if he claims at least one element of every minimal winning set. Since a monotone increasing family and the family of its minimal elements uniquely determine each other, often, when there is no risk of confusion, we use $\mathcal{F}$ for the family of minimal winning sets as well. A classical example of the Maker-Breaker setting is the popular board-game Hex.

The study of Maker-Breaker games on the edges of a (complete) graph was initiated by Lehman [60] who, in particular, proved that in the $(1, 1)$ game on K_n, Maker can easily build a spanning tree (by "easily" we mean that he can do so within $n - 1$ moves). Chvátal and Erdős [30] suggested to "even out the odds" by giving Breaker more power, that is, by increasing his bias (note that, in Maker-Breaker games, a larger bias cannot harm a player; we refer to this property as *bias monotonicity*). They determined that the $(1, b)$ game $\mathcal{T}_n$ of "Connectivity", where the family $\mathcal{T}_n$ of minimal winning sets consists of the edge-sets of all spanning trees of K_n, is won by Maker even when the bias b of Breaker is as large as $cn/\log n$ for some small constant $c > 0$, while for some large enough constant $C > 0$, Breaker wins if his bias is at least $Cn/\log n$. They also showed that the $(1, 1)$ game $\mathcal{H}_n$ of "Hamiltonicity", where Maker's goal is to build a Hamiltonian cycle (the family $\mathcal{H}_n$ of minimal winning sets consists of the edge-sets of all Hamiltonian cycles of K_n), is won by Maker for sufficiently large n. They also conjectured that in fact Maker can win the $(1, b)$ Hamiltonicity game for some b that tends to infinity with n. This was proved by Bollobás and Papaioannou [27], who showed that Maker wins Hamiltonicity against a bias of $O(\log n/\log\log n)$. Beck [9] gave a winning strategy for Maker against a bias of $\left(\frac{\log 2}{27} - o(1)\right)\frac{n}{\log n}$, thus determining the correct order of magnitude of the *threshold bias*. Very recently, Krivelevich and Szabó [58] were able to improve Beck's bound to $(\log 2 - o(1))\,\frac{n}{\log n}$.

Following [42] it will be convenient to introduce the following notation. For a family $\mathcal{F}$ of winning sets, let $b_{\mathcal{F}}$ be the non-negative integer for which Breaker has a winning strategy in the $(1, b)$ game $\mathcal{F}$ if and only if $b \geq b_{\mathcal{F}}$. Note that $b_{\mathcal{F}}$ is well-defined for any (monotone increasing) family $\mathcal{F}$ (unless $\mathcal{F}$ contains a hyperedge of size at most one). By the above, $b_{\mathcal{T}_n} = \Theta(n/\log n)$ and $b_{\mathcal{H}_n} = \Theta(n/\log n)$.

Observe the intriguing relation between the threshold biases of the Connectivity and Hamiltonicity games and the threshold probability at which the random graph $G(n, p)$ first possesses these properties. The number of edges of the random graph $G(n, p)$ around this threshold $p = \log n/n$ has the same order of magnitude as the number of edges Maker has in his graph after playing against the threshold bias $b_{\mathcal{T}_n}$ or $b_{\mathcal{H}_n}$. This phenomenon (known as the Erdős paradigm) was pointed out by Beck [9], where this observation is attributed to Erdős (indeed the first such result appeared already in [30]). Since then, several other games have been found to have a threshold bias which is closely linked to a meaningful random graph threshold related to that particular game (see, e.g. [11, 16, 20, 21, 70]). Note, that the asymptotic values of the threshold biases $b_{\mathcal{T}_n}$ and $b_{\mathcal{H}_n}$ are not known in general. It can still turn out that these threshold biases are asymptotically equal to the inverse of the corresponding sharp thresholds for the appropriate properties of random graphs.

In the current chapter we investigate this connection further and find three more instances where such an intuition proves to be correct. Let $\mathcal{NP}_n$, $\mathcal{NC}_n^k$, $\mathcal{M}_n^t$ consist of the edge-sets of all non-planar graphs on n vertices, the edge-sets of all non-k-colorable graphs on n vertices, and the edge-sets of all graphs on n vertices containing a K_t-minor (a graph G contains an H-minor if H can be obtained from G by the deletion and contraction of some of its edges), respectively. Obviously, all three families are monotone increasing.

In the game $\mathcal{NP}_n$, which we call the "planarity game" (although perhaps "non-planarity" would have been a more appropriate name), Maker's goal is to claim all edges of a non-planar graph. In Theorem 2.1 we show that the corresponding threshold bias $b_{\mathcal{NP}_n}$ is asymptotically $n/2$. Note that this result is a consequence of the result for the minor game we obtain in Theorem 2.7, as the presence of a K_5-minor guarantees non-planarity. Nevertheless, we include an alternative, direct proof of Theorem 2.1 which approaches planarity from a different angle, and, in our opinion, is more illustrative.

Coming back to the relation between the probability thresholds of random graphs and the thresholds for the game bias, we note that the threshold probability $p_{\mathcal{NP}_n}$ of non-planarity in the random graph is $1/n$, that is, $p_{\mathcal{NP}_n} = \Theta(1/b_{\mathcal{NP}_n})$. However, the number of edges of $G(n, p_{\mathcal{NP}_n})$ is concentrated around *one half* of the number of edges in Maker's graph in the $(1, b_{\mathcal{NP}_n}, \mathcal{NP}_n)$ game (the non-planarity threshold is sharp, so such a comparison makes sense).

In the "k-colorability game", the family of winning sets is $\mathcal{NC}_n^k$, that is, Maker wins the game if he claims a non-k-colorable graph. In Theorem 2.3 we show that the threshold bias is linear in n for a fixed k by establishing that there are absolute constants c_1, c_2 such that, $c_1 \frac{n}{k \log k} \leq b_{\mathcal{NC}_n^k} \leq c_2 \frac{n}{k \log k}$ for every k. For the special case $k = 2$, that is, the bipartite game, a more accurate result was proved by Bednarska and Pikhurko [23]. They showed that $(1 - 1/\sqrt{2} - o(1))n \leq b_{\mathcal{NC}_n^2} \leq \lceil n/2 \rceil - 1$. Again, one can compare these results with the corresponding random graph threshold $p_{\mathcal{NC}_n^k} = \frac{2k \log k}{n}$ for non-k-colorability and find the reciprocal relation.

Finally, we turn to a more general setting. In the K_t-minor game the family of winning sets is $\mathcal{M}_n^t$; hence, Maker wins the game if he is able to claim the edges of some K_t-minor of K_n. This game is in some sense a generalization of both the planarity and the colorability games we have discussed. Indeed, Wagner's Theorem gives a full description of planarity via the language of forbidden minors. Furthermore, if a graph is not r-colorable then it contains a K_s-minor for $s = r/(c\sqrt{\log r})$ where c is an absolute constant (see [54], [73]), and the famous and long standing Hadwiger conjecture asserts that in fact it contains a K_r-minor. The opposite implication is trivially false as for every positive integer n, the complete bipartite graph $K_{n,n}$ admits a K_n-minor.

In Theorem 2.7 we prove that the corresponding threshold bias $b_{\mathcal{M}_n^t}$ is asymptotically $n/2$ for every $3 \leq t \leq c\sqrt{n/\log n}$, for an appropriate constant c. Note that for the case $t = 3$, an accurate threshold bias follows from a result of Bednarska and Pikhurko [22], as a graph is K_3-minor free if and only if it is a forest (see the exact statement in Theorem 2.2 in Section 2.2).

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. All of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout this chapter, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [31].

The rest of this chapter is organized as follows: in Section 2.2 we discuss the Maker-Breaker planarity game, in Section 2.3 we discuss the Maker-Breaker colorability game and in Section 2.4 we discuss the Maker-Breaker K_t-minor game. Finally, in Section 2.5 we present some open problems.

2.2 The Maker-Breaker planarity game

The following theorem states that the threshold bias at which Maker's win turns into a Breaker's win in the planarity game is "around" $n/2$.

Theorem 2.1

$$b_{\mathcal{NP}_n} = \frac{n}{2} - o(n).$$

Proof: Let $b \geq n/2$. The existence of a winning strategy for Breaker in the planarity game is an easy consequence of the following result of Bednarska and Pikhurko.

Theorem 2.2 *[22, Corollary 10] Suppose that CycleMaker and CycleBreaker select respectively* 1 *and* b *edges of* K_n *and CycleMaker wins if he builds a cycle. Then CycleMaker has a winning strategy (no matter who starts) if and only if* $b < \lceil n/2 \rceil$.

The assertion of Theorem 2.2 implies that with the bias $b \geq n/2$, Breaker can prevent Maker from building a cycle. It follows that at the end of the game Maker's graph will be a forest which is obviously planar.

Next, let $0 < \varepsilon < 1/3$ (the restriction $\varepsilon < 1/3$ is technical) and let $b \leq (1/2 - \varepsilon)\, n$, where $n = n(\varepsilon)$ is sufficiently large. We will provide Maker with a strategy for building Maker a non-planar graph. Let $\alpha = \frac{2\varepsilon}{1-2\varepsilon}$ and let $\alpha_n = \alpha_n(\varepsilon)$ be the real number satisfying the equation

$$(1 + \alpha_n)n = \frac{\binom{n}{2}}{(\frac{1}{2} - \varepsilon)n + 1}.$$

Then $\lim_{n\to\infty} \alpha_n = \alpha$. Let m_n denote the number of edges that Maker will claim by the end of the game on K_n. We have $m_n - (1 + \frac{\alpha}{2})n = \Omega(n)$.

Let $k = k(\varepsilon)$ be the smallest positive integer such that

$$\left(1 + \frac{\alpha}{2}\right) > \frac{k}{k-2}.$$

Maker's goal is to avoid cycles of length smaller than k, which we will call "short cycles", during the first $\left(1 + \frac{\alpha}{2}\right) n$ moves. If he succeeds, Maker's graph will at that point of the game have

$$\left(1 + \frac{\alpha}{2}\right) n > \frac{k}{k-2} n$$

edges and girth at least k. But, it is well-known that a planar graph with girth at least k cannot have more than $\frac{k}{k-2}(n-2)$ edges. Hence, Maker's graph will already be non-planar, and he will win no matter how the game continues.

It remains to show that Maker can indeed avoid claiming a short cycle during the first $(1 + \frac{\alpha}{2})n$ moves. His strategy is the following. For as long as possible he claims edges (u, v) that satisfy the following two properties:

(a) (u, v) does not close a short cycle;

(b) the degrees of both u and v in Maker's graph are less than $n^{1/(k+1)}$.

It suffices to prove that when this is no longer possible, that is, every remaining unclaimed edge violates either (a) or (b), Maker has already claimed at least $(1 + \frac{\alpha}{2})n$ edges.

Every edge that violates property (b) must have at least one endpoint of degree $n^{1/(k+1)}$ in Maker's graph. Since Maker's graph at any point of the game contains at most $(1 + \alpha)n$ edges, there are at most $2(1+\alpha)n^{1-1/(k+1)}$ vertices of degree at least $n^{1/(k+1)}$. Therefore, the number of edges that violate property (b) is at most

$$n \cdot 2(1 + \alpha)n^{1-1/(k+1)} = o(n^2).$$

For any fixed $s < k$ and every vertex v, the number of paths of length s that have v as one endpoint is at most Δ^s, where Δ is the maximum degree in Maker's graph. If we assume that property (b) has not been violated, then $\Delta \leq n^{1/(k+1)}$. Therefore, there are at most

$$n \cdot \sum_{s=3}^{k-1} n^{s/(k+1)} = o(n^2)$$

edges that close a short cycle.

Thus, the total number of edges that violate (a) or (b) if claimed by Maker, is $o(n^2)$. On the other hand, after $(1 + \frac{\alpha}{2})n$ moves have been played, the number of unclaimed edges is $\Theta(n^2)$. Hence, in the first $(1 + \frac{\alpha}{2})n$ moves Maker can claim edges that satisfy properties (a) and (b), which means that he does not claim a short cycle. This completes the proof of the theorem. $\square$

2.3 The Maker-Breaker k-colorability game

The following theorem states that the threshold bias at which Maker's win turns into a Breaker's win in the k-colorability game, where k is fixed and n is sufficiently large, is of order n. This is true for every $k \geq 2$. However, for convenience, and since the case $k = 2$ was treated in [23], we will assume that $k \geq 3$.

Theorem 2.3 *For every $k \geq 3$ there exist constants s_k and s'_k such that*

$$s'_k n \leq b_{\mathcal{NC}_n^k} \leq s_k n,$$

where $s_k \sim \frac{2}{k \log k}$ as $k \to \infty$, and $s'_k \sim \frac{\log 2}{2k \log k}$ as $k \to \infty$.

Proof: Assume first that $b \leq \frac{n}{ck \log k}$. We will provide Maker with a strategy for building a non-k-colorable graph. Maker's goal will be to prevent Breaker from building a clique of size $\lceil n/k \rceil$, and this is enough to ensure his win. Indeed, Maker's graph is surely not k-colorable if it does not admit an independent set of size $\lceil n/k \rceil$.

Let $\mathcal{F}$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are the $\lceil n/k \rceil$-cliques of K_n. We name the players of the $(b, 1, \mathcal{F})$ game, CliqueMaker and CliqueBreaker. As was mentioned before, Maker wins the k-colorability game if he claims a vertex in every hyperedge of $\mathcal{F}$, that is, if he is able to win the $\mathcal{F}$ game as CliqueBreaker. We will use Beck's criterion, which is applicable to any Maker/Breaker-type game.

Theorem 2.4 *[6, Theorem 1] If*

$$\sum_{D \in \mathcal{H}} (1 + q)^{-|D|/p} < \frac{1}{q + 1},$$

then Breaker wins the (p, q) game $\mathcal{H}$.

We have

$$\sum_{D\in\mathcal{F}} 2^{-|D|/b} \leq \binom{n}{\lceil n/k\rceil} 2^{-\binom{\lceil n/k\rceil}{2}/b} \leq (ek)^{\lceil n/k\rceil} 2^{-\binom{\lceil n/k\rceil}{2}/b}$$
$$\leq 2^{\frac{n\log_2 e}{k}+\frac{n\log_2 k}{k}+\log_2(ek)-\frac{cn^2 k\log k}{2k^2 n}+\frac{nck\log k}{2kn}} = o(1).$$

Hence, Maker can win the k-colorability game.

Assume now that $b \geq s_k n$, where s_k is a constant depending on k that will be determined later. We will provide Breaker with a strategy, to force Maker into building a k-colorable graph. We will make use of the following two theorems.

Theorem 2.5 *[20, Theorem 1] For every graph G which contains at least three non-isolated vertices there exist positive constants c and n_0 such that, playing the $(1, q)$ game on K_n, G-Breaker can prevent G-Maker from building a copy of G provided that $n > n_0$ and $q > cn^{1/m_2(G)}$, where*

$$m_2(G) = \max_{\substack{H\subseteq G\\ v(H)\geq 3}} \frac{e(H)-1}{v(H)-2}.$$

Theorem 2.6 *[51, Corollary 1.2] If G is a graph with maximum degree Δ and girth at least 5, then*

$$\chi(G) \leq (1+\nu(\Delta))\frac{\Delta}{\log\Delta},$$

where $\nu(\Delta) \to 0$ as $\Delta \to \infty$.

Let Δ_0 be the maximal value of Δ for which

$$(1+\nu(\Delta))\frac{\Delta}{\log\Delta} \leq k$$

(if no such Δ_0 exists or if $\Delta_0 < 2$, then we take $s_k = 1/2$ and so Breaker wins by Theorem 2.2).

Since $\nu(\Delta) \to 0$ as $\Delta \to \infty$, we have that $\Delta_0 \sim k\log k$ as $k \to \infty$. Breaker's goal will be to force Maker to build a graph with maximum degree at most Δ_0 and girth at least 5. By Theorem 2.6 Maker's graph will then be k-colorable. In each move, Breaker will use $c_3 n^{1/2}$ of his edges to prevent Maker from building a triangle (recall that $m_2(C_3) = 2$), and $c_4 n^{2/3}$ of his edges to prevent Maker from building a cycle of length 4 ($m_2(C_4) = 3/2$), where c_3 and c_4 are the constants whose existence is guaranteed by Theorem 2.5. Breaker will use all of his remaining $b' := b - c_3 n^{1/2} - c_4 n^{2/3} = (1-o(1))b$ edges to make sure that the maximum degree in Maker's graph does not surpass Δ_0. Hence, if Maker claims the edge (u, v), then Breaker will claim $\frac{1}{2}b'$ edges incident with u and $\frac{1}{2}b'$ edges incident with v (if there are only $r < \frac{1}{2}b'$ unclaimed edges incident with u or v, then Breaker will claim all of them and additional $\frac{1}{2}b' - r$ arbitrary unclaimed edges). It follows that the maximum degree in Maker's graph will be at most

$$1 + \frac{n-1}{b'/2} \leq 1 + \frac{2n}{(1-o(1))b} \leq 1 + \frac{2}{s_k} + o(1),$$

where the $o(1)$ term tends to zero as n tends to infinity. Therefore, if $s_k = \lceil \frac{2}{\Delta_0 - 1.5} \rceil$, then Maker's graph will have maximum degree at most Δ_0. Hence, Breaker can force Maker to build a graph with maximum degree at most Δ_0 and girth at least 5, and thus he can win. Note that s_k, defined this way, satisfies $s_k \sim \frac{2}{k \log k}$ as $k \to \infty$. This concludes the proof of the theorem. □

2.4 The Maker-Breaker minor game

In the Maker-Breaker version of the game, Maker's goal is to build a graph that contains a K_t-minor. The following theorem states that the threshold bias at which Maker's win turns into a Breaker's win in the K_t-minor game for every $3 \leq t \leq c\sqrt{n/\log n}$, for an appropriate constant c, is asymptotically $n/2$.

Theorem 2.7 *For every fixed $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$, such that if n is sufficiently large and $b \leq (1-\varepsilon)n/2$, then Maker has a winning strategy for the $(1, b)$ game $\mathcal{M}_n^t$ for every $t < C\sqrt{n/\log n}$.*

As a corollary we have the asymptotics of $b_{\mathcal{M}_n^t}$ for arbitrary fixed t.

Corollary 2.8 *Let $t \geq 3$ be a positive integer. Then*

$$b_{\mathcal{M}_n^t} = \frac{n}{2} - o_t(n).$$

The lower bound in the corollary follows from Theorem 2.7, while if $b \geq n/2$, then Breaker, as in the proof of Theorem 2.1, can force Maker to build a forest, which does not contain a K_t-minor for any $t \geq 3$.

In the proof of Theorem 2.7 we will use the following result of Kostochka and of Thomason.

Theorem 2.9 *([54], [73]) There exists a constant c' such that every graph of average degree at least $c'r\sqrt{\log r}$ admits a K_r-minor.*

Proof of Theorem 2.7 Assume that $b \leq (1-\varepsilon)n/2$. We will provide Maker with a strategy for building a graph that admits a large minor. Maker's strategy is divided into two stages. In the first stage that lasts exactly $m-1$ (where m is to be determined later) rounds, Maker builds a tree $T = (V, E)$ that satisfies the following properties:

1. $|V| = m \geq \varepsilon n$,

2. the degree of every $u \in V$ is at most 3,

3. there remain at least $\varepsilon^2 n^2/3$ unclaimed edges with both endpoints in V.

Maker's strategy for building such a tree is very simple: he starts by claiming an arbitrary edge and then, for as long as possible he claims a previously unclaimed edge (u, v) such that, in his current graph, u has degree 1 or 2 and v is isolated; such an edge, when exists, is chosen arbitrarily. Clearly this results in a tree with maximum degree at most 3. Now, assume, that using this strategy, Maker was able to build a tree on m vertices but could not extend it to a tree on $m+1$ vertices (while maintaining the maximum degree criterion). This means that every edge (u, v) such that, in Maker's graph, u has degree 1 or 2 and v is isolated, must have been claimed by Breaker. Since Maker's graph is a tree, at least half its vertices have degree at most 2 and so Breaker must have claimed at least $\frac{m}{2}(n-m)$ edges. But at this point, Breaker has at most $m(1-\varepsilon)n/2$ edges entailing $m \geq \varepsilon n$. Furthermore, the number of edges with both endpoints in V that Breaker could have claimed is at most $m(1-\varepsilon)n/2 - \frac{m}{2}(n-m) = \frac{m^2}{2} - \frac{m\varepsilon n}{2}$. It follows that there must be at least $\frac{m\varepsilon n}{2} - \frac{3m}{2} \geq \varepsilon^2 n^2/3$ unclaimed edges with both endpoints in V. This ends the first stage.

Before claiming edges in the second stage, Maker would like to partition T into roughly $\sqrt{n}$ connected components of roughly the same size. The following result of Krivelevich and Nachmias asserts that he can.

Lemma 2.10 *[55, Proposition 4.5] Let $G = (V, E)$ be a connected graph on r vertices with maximum degree at most k. Then for every positive integer l, there exist pairwise disjoint sets $V_1, \ldots, V_s \subseteq V$, with the following properties:*

1. *$lk \leq |V_i| \leq lk^2$ for every $1 \leq i \leq s$.*
2. *$\sum_{i=1}^{s} |V_i| \geq r - lk$.*
3. *$G[V_i]$ is connected for every $1 \leq i \leq s$.*

Using Lemma 2.10 with $r = m$, $l = \varepsilon^{-1}\sqrt{m}$ and $k = 3$ we conclude that at least $m - 3\varepsilon^{-1}\sqrt{m}$ of the vertices of T can be partitioned into parts $V_1, \ldots, V_s$ such that $3\varepsilon^{-1}\sqrt{m} \leq |V_i| \leq 9\varepsilon^{-1}\sqrt{m}$ and $T[V_i]$ is connected for every $1 \leq i \leq s$. Note that $\varepsilon\sqrt{m}/10 \leq s \leq \varepsilon\sqrt{m}/3$.

A pair (V_i, V_j) will be called *good* if there are at least $b+1$ unclaimed edges (u, v) where $u \in V_i$ and $v \in V_j$. We claim that at least an $\varepsilon^2/20$ fraction of the total number of pairs is good. Indeed, assume for the sake of contradiction that there are less than $\varepsilon^2\binom{s}{2}/20$ good pairs, then there are at most

$$3m\varepsilon^{-1}\sqrt{m} + \sum_{i=1}^{s}\binom{|V_i|}{2} + b\binom{s}{2} + \varepsilon^2\binom{s}{2}(9\varepsilon^{-1}\sqrt{m})^2/20$$

unclaimed edges in V (the first term stands for the edges incident with vertices outside $\bigcup_{i=1}^{s} V_i$, the second term stands for edges inside the V_i's, the third term stands for unclaimed edges that might remain between any pair, even if it is not good, and the fourth term stands for edges between good pairs).For sufficiently large n this is strictly less than $\varepsilon^2 n^2/3$; this contradicts Maker's strategy for the first stage.

For every good pair (V_i, V_j), let $A_{i,j}$ be any set of $b+1$ unclaimed edges with one endpoint in V_i and the other in V_j. Trivially, by simply not claiming more than one edge from any $A_{i,j}$ (and not claiming edges outside the $A_{i,j}$'s for as long as possible), Maker can claim an edge of $A_{i,j}$ for at least half of the good pairs (V_i, V_j). In the second stage, Maker will use this strategy; denote Maker's graph at the end of the second stage by H. Consider the graph $\tilde{H}$ on the vertex set $V(\tilde{H}) = \{V_1, \ldots, V_s\}$, where (V_i, V_j) is an edge iff Maker has claimed an edge (x, y) such that $x \in V_i$ and $y \in V_j$. The average degree in $\tilde{H}$ is at least

$$\frac{\varepsilon^2 \binom{s}{2}/20}{s} \geq \varepsilon^3 \sqrt{m}/400 \geq \varepsilon^4 \sqrt{n}/400$$

and so by Theorem 2.9 it admits a K_t-minor for $t = C\sqrt{n/\log n}$, for an appropriate constant C. Since $T[V_i]$ is connected for every $1 \leq i \leq s$, H admits the same minor and the proof of Theorem 2.7 is complete. □

2.5 Concluding remarks and open problems

Threshold biases. In Section 2.3 it was proved that $b_{\mathcal{NC}_n^k} = \Theta(n)$. We believe that in fact the following stronger statement holds.

Conjecture 2.11 *There is a constant c, such that for every $k \geq 3$ and sufficiently large n we have*

$$b_{\mathcal{NC}_n^k} = \frac{(c + o(1))}{k \log k} n.$$

For both the planarity and the K_t-minor Maker/Breaker games, the second order terms are unknown; this is worth studying, in particular the dependence of the threshold $b_{\mathcal{M}_n^t}$ on t.

Chapter 3

Avoider-Enforcer games

3.1 Introduction

Let p and q be positive integers and let $\mathcal{H}$ be any hypergraph. In a $(p, q, \mathcal{H})$ Avoider-Enforcer game two players, called Avoider and Enforcer, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Avoider selects p vertices per move and Enforcer selects q vertices per move. The game ends when every vertex has been claimed by one of the players. Avoider loses if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Enforcer loses. The integer p is called the bias of Avoider, and q is called the bias of Enforcer. We assume that Avoider starts the game unless explicitly stated otherwise, although the identity of the player who makes the first move is usually irrelevant due to the asymptotic nature of our results.

The hypergraph $\mathcal{H}$ is sometimes referred to as the *game* (without mentioning the biases). We call the game $(p, q, \mathcal{H})$ an *Avoider's win* (*Enforcer's win*) if Avoider (Enforcer) has a winning strategy in the $(p, q, \mathcal{H})$ game. It is not hard to see that every game $(p, q, \mathcal{H})$ is either an Avoider's win or an Enforcer's win, but not both.

Arguably, the goals of the players in Avoider-Enforcer games are not the most natural ones. The goal of Avoider is defined through a negation, that is, he wins if he does *not* occupy any member of $\mathcal{H}$. The variant of these games with "positive" goals is indeed much more thoroughly studied. In a Maker-Breaker type game, the player called Maker wins if he *does* occupy all the vertices of some member of $\mathcal{H}$; otherwise the other player (Breaker) wins. However, we argue that Avoider-Enforcer games are as natural. First of all any game in which the goal of one of the players is to build a graph which satisfies some monotone decreasing property, is an Avoider-Enforcer game (that player is Avoider). Furthermore, in discrepancy-type games (see [38], [43]) the goal of one of the players is to claim some fixed percentage (not more and not less) of every winning set, hence he plays as both Breaker and Avoider.

Putting aside a few scattered results, the theory of Maker-Breaker games started with a general criterion of Erdős and Selfridge [36] for Breaker's win in the $(1, 1)$ game. Subsequently, Beck started a systematic study of Maker-Breaker games with a bias ([5, 6, 10, 12, 13, 14, 15] to name just a few). In particular in [6] he proved the following generalization of the Erdős-Selfridge criterion: If

$$\sum_{D\in\mathcal{H}} (1+q)^{-|D|/p} < \frac{1}{1+q} \tag{3.1}$$

then Breaker has a winning strategy for the $(p, q, \mathcal{H})$ Maker-Breaker game.

Lu [61] proved that an identical criterion guarantees Avoider's win in the $(1, 1)$ game. One can (somewhat naively) assume that the theory of Avoider-Enforcer games is very similar to that of Maker-Breaker games, and that criterion (3.1) guarantees a winning strategy for Avoider for arbitrary p and q. As it turns out things are much more complicated, as the case of $(1, 1)$ games is somewhat special and hides the difficulties that arise in the biased games.

Further thought reveals that the differences between Maker-Breaker and Avoider-Enforcer games go much deeper. Without giving it much thought, one expects (and rightly so) that Maker's win and Breaker's win will be appropriately monotone in the bias. That is, if for example Maker wins the $(1, 1)$ game, then he will also win the $(2, 1)$ game on the same hypergraph. The simple-minded reason for this is that "more occupied vertices cannot hurt Maker, and in fact, might even help him". This is indeed true and will be discussed further in Section 3.9. Now, it is equally plausible to assume that in case Enforcer wins the $(1, 1)$ game, he will also win the $(2, 1)$ game, since "less occupied vertices cannot hurt Enforcer". It turns out that this intuition fails. For example it is possible to give examples of $(1, q)$ (resp. $(p, 1)$) Avoider-Enforcer games which are won by Enforcer iff q (resp. p) is of a certain parity. We explore these issues of monotonicity in Section 3.9.

Despite these major differences, one is able to adapt Beck's argument to some extent and to provide an analogous criterion for Avoider's win.

Theorem 3.1 *[42, Theorem 1.1] If Avoider is the last player (that is, the player to make the last move) and*

$$\sum_{D\in\mathcal{H}} \left(1+\frac{1}{p}\right)^{-|D|} < \left(1+\frac{1}{p}\right)^{-p}$$

then Avoider wins the $(p, q, \mathcal{H})$ game for every $q \geq 1$.
If Enforcer is the last player then the above sufficient condition can be relaxed to

$$\sum_{D\in\mathcal{H}} \left(1+\frac{1}{p}\right)^{-|D|} < 1.$$

Note, that though we assume for convenience that Avoider starts the game, the assertion of Theorem 3.1, holds also when Enforcer starts the game.

Our criterion does not take into account the value of q, so it is unlikely to be best possible. For any constant value of q, however, we show that the criterion is "not far" from being best possible. Beck [6] proved that his sufficient condition (3.1) for Maker-Breaker games is best possible by building explicitly an infinite family of hypergraphs $\mathcal{H}$ such that equality holds in (3.1) and Maker has a winning strategy for the corresponding game. We think that the problem of finding a useful, and possibly "best possible" criterion for Avoider's win when $q > 1$, is one of the most interesting open problems of the topic.

Theorem 3.2 *For every positive integers p and q there are infinitely many hypergraphs $\mathcal{H}$ such that $\sum_{D\in\mathcal{H}}(1+\frac{1}{p})^{-|D|} \le \binom{p+q-1}{q-1}\frac{p+1}{2}$, and yet Enforcer wins the $(p,q,\mathcal{H})$ game.*

Note that $\binom{p+q-1}{q-1}\frac{p+1}{2}$ is polynomial in p for every fixed q.

In this chapter we study more closely six quite natural Avoider-Enforcer games (some of them were studied by other researchers, mostly the unbiased case (see e.g. [17, 61, 62, 63, 64])): "connectivity", "perfect matching", "hamiltonicity", "non-planarity", "non-colorability" and "minor". Let $\mathcal{T}_n$ be the family of all spanning trees, $\mathcal{M}_n$ be the family of all perfect matchings (here we assume of course that n is even), $\mathcal{H}_n$ be the family of all Hamilton cycles, $\mathcal{NP}_n$ be the family of all non-planar subgraphs, $\mathcal{NC}_n^k$ be the family of all non-k-colorable subgraphs and $\mathcal{M}_n^t$ be the family of all K_t minors in the complete graph K_n on n vertices.

It will be convenient to introduce the following notation. For a hypergraph $\mathcal{B}$ we define $b_{\mathcal{B}}^-$ to be the largest integer such that Enforcer can win $(1,b,\mathcal{B})$ for every $b \le b_{\mathcal{B}}^-$, and $b_{\mathcal{B}}^+$ to be the smallest integer such that Avoider can win $(1,b,\mathcal{B})$ for every $b > b_{\mathcal{B}}^+$. Except for certain degenerate cases, $b_{\mathcal{B}}^-$ and $b_{\mathcal{B}}^+$ always exist and satisfy $b_{\mathcal{B}}^- \le b_{\mathcal{B}}^+$. However, as was indicated above, we do not know in general that $b_{\mathcal{B}}^- = b_{\mathcal{B}}^+$, that is, we do not know whether a well-defined threshold bias exists. In case $b_{\mathcal{B}}^- = b_{\mathcal{B}}^+$, we denote this number by $b_{\mathcal{B}}$ and call it the *threshold bias* of the game $\mathcal{B}$.

For Maker-Breaker games a similar threshold bias, at which a Maker's win turns into a Breaker's win could be defined and *does exist* for all hypergraphs. It was proved by Chvátal and Erdős [30] and by Beck [9] that the threshold bias for all three Maker-Breaker games $\mathcal{T}_n, \mathcal{M}_n$, and $\mathcal{H}_n$ is of order $n/\log n$. The remaining three games $\mathcal{NP}_n$, $\mathcal{NC}_n^k$ and $\mathcal{M}_n^t$ were not studied before.

As a first application of Theorem 3.1 we consider the perfect matching game $(1,q,\mathcal{M}_n)$.

Theorem 3.3 *Enforcer has a winning strategy in $(1,q,\mathcal{M}_{2n})$ if $q < \frac{n}{(2+o(1))\log_2 n}$ and n is sufficiently large. Thus,*

$$b_{\mathcal{M}_n}^- = \Omega\left(\frac{n}{\log n}\right).$$

Although it looks plausible we do not know whether the perfect matching game is monotone. Moreover, even if a threshold does exists, we do not know whether it is of order $n/\log n$.

Next, we will use Theorem 3.1 to prove a sufficient condition for Enforcer to win the Hamilton cycle game $(1,q,\mathcal{H}_n)$. For every positive integer k, we denote by $\log^{(k)} n$ the k-fold natural logarithm of n, (that is, $\log^{(1)} n = \log n,\ \log^{(2)} n = \log\log n,$ etc).

Theorem 3.4 *Enforcer has a winning strategy in $(1,q,\mathcal{H}_n)$ if $q < \frac{n \log 2 \log^{(4)} n}{8500 \log n \log^{(3)} n}$ and n is sufficiently large. Thus,*

$$b_{\mathcal{H}_n}^- = \Omega\left(\frac{n}{\log n}\cdot\frac{\log^{(4)} n}{\log^{(3)} n}\right).$$

Note that, using similar methods, Krivelevich and Szabó [58] were able to prove very recently that

$$b^-_{\mathcal{H}_n} = \Omega\left(n/\log n\right).$$

In [30] and [6] it is shown that the threshold bias for the $(1, q)$ Maker-Breaker connectivity game is between $(\log 2 - \varepsilon)\frac{n}{\log n}$ and $(1 + \varepsilon)\frac{n}{\log n}$ for every $\varepsilon > 0$. It is also suggested there that the order of magnitude $n/\log n$ is very reasonable for this problem as at the end of the game Maker will have about $\frac{1}{2}n\log n$ edges which is the threshold for the connectivity of a random graph $G(n, m)$ (see e.g. [50] for background on random graphs). Hence we find it somewhat surprising that this insight fails badly for the Avoider-Enforcer connectivity game. We would also like to stress that this is the only case where we could establish the monotonicity of a game of interest.

Theorem 3.5 *Avoider wins the $(1, q)$ connectivity game $\mathcal{T}_n$ iff at the end of the game he has at most $n-2$ edges. In particular the threshold $b_{\mathcal{T}_n}$ exists for every n. We have $b_{\mathcal{T}_n} = \left\lfloor \frac{n}{2} \right\rfloor - 1$, except when n is odd and Avoider starts the game, in which case $b_{\mathcal{T}_n} = \left\lfloor \frac{n}{2} \right\rfloor$.*

To a certain extent we can generalize the assertion of Theorem 3.5 to the "k-edge connectivity game", in which Avoider loses iff he builds a k-edge connected spanning subgraph of K_n. Unfortunately, if $k > 1$, we do not know the exact bias, nor do we know whether it exists, that is, whether the corresponding game is monotone.

Theorem 3.6 *Playing on K_n, if $q \le \frac{n}{2k} - 1$ then Enforcer wins the k-edge connectivity game. If $q \ge \frac{n}{k}$ then Avoider wins the k-edge connectivity game.*

In [62], Lu considered the $(1, 1)$ Avoider-Enforcer game on the edges of K_n, in which Enforcer's goal is to force Avoider to build as many pairwise edge disjoint spanning trees as possible. Clearly $\lfloor n/4 \rfloor$ is an upper bound. Lu proved that for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that if $n \ge n_0$ then, playing on K_n, Enforcer can force Avoider to build $(1 - \varepsilon)n/4$ pairwise edge disjoint spanning trees. We prove that the trivial upper bound is also a lower bound.

Theorem 3.7 *For every positive integer n, playing on K_n, Enforcer can force Avoider to build $\lfloor n/4 \rfloor$ pairwise edge disjoint spanning trees.*

All of the three above results are relatively easy consequences of the following theorem.

Theorem 3.8 *If G contains $q + 1$ pairwise edge disjoint spanning trees, then Enforcer, as first or second player, wins the $(1, q)$ connectivity game on G.*

Observe that the case $q = 1$ of the above theorem can be considered as the Avoider-Enforcer analog of the celebrated Lehman's criterion [60] for Maker's win in connectivity games.

In Theorem 3.10 we prove that Avoider can keep his graph planar against any bias which is larger than $2n^{5/4}$, whereas Enforcer wins when playing with bias at most $\frac{n}{2} - o(n)$. As in the case of the Maker-Breaker planarity game, the second part of this result is a direct consequence of our result for the minor game presented in Theorem 3.13. Nevertheless, we include an alternative, direct proof which relies on other properties of planar graphs. We believe it is more illustrative and gives more insight in the course of the game.

In Theorem 3.12 we show that playing against a bias of at least $2kn^{1+1/(2k-3)}$, Avoider can keep his graph k-colorable, whereas Enforcer wins if his bias is at most $\frac{n}{ck \log k}$.

In the Avoider-Enforcer version of the K_t-minor game, Avoider's task is to build a K_t-minor free graph. As in the planarity and colorability games, Avoider's goal in this game is very natural as many graph-theoretic properties can be expressed in a "forbidden minor" fashion. In Theorem 3.13 we prove that playing with bias at most $n/2 - o(n)$, Enforcer can make sure that by the end of the game Avoider will claim the edges of some K_t-minor of K_n, where t is arbitrarily large but fixed.

Moreover, we prove that playing with a linear bias, Enforcer can make Avoider build a graph that admits a K_t minor for t which is as large as $c\sqrt{n/\log n}$.

In order to overcome the non-monotonicity of Avoider-Enforcer games and, as a result, the lack of a well defined threshold bias, we offer a change in the rules of Avoider-Enforcer games. The rules stay the same as in the usual case, except that both players are allowed to claim more elements per turn. That is, in a $(p, q, \mathcal{H})$ game, Avoider claims **at least** p elements of $\mathcal{H}$ per turn (instead of **exactly** p) and Enforcer claims **at least** q elements of $\mathcal{H}$ per turn. It is easy to see that Avoider-Enforcer games with these rules are monotone and thus have a threshold bias. In Section 3.10, we study the effect of these rules on Avoider-Enforcer games and, in particular, find very tight bounds on the threshold bias of several interesting games. There is a surprising difference between these and the bounds we have obtained for "regular" Avoider-Enforcer games.

Throughout this chapter, for the sake of simplicity and clarity of presentation, we omit floor and ceiling signs whenever these are not crucial. All logarithms are natural unless explicitly stated otherwise. Our graph-theoretic notation is standard and follows that of [31]. In particular, for a graph $G = (V, E)$ and a set $A \subseteq V$, let

$$N_G(A) = \{u \in V : \exists w \in A, (u, w) \in E\}$$

be the neighborhood of A in G. Often, when there is no risk of confusion, we abbreviate $N_G(A)$ to $N(A)$.

The rest of this chapter is organized as follows: In Section 3.2 we prove Theorems 3.1 and 3.2. In Section 3.3 we prove Theorem 3.3, in Section 3.4 we prove Theorem 3.4, in Section 3.5 we prove Theorem 3.8 and then derive Theorems 3.5, 3.6 and 3.7. In Section 3.6 we prove Theorem 3.10, in Section 3.7 we prove Theorem 3.12, and in Section 3.8 we prove Theorems 3.13 and 3.16. In Section 3.9 we discuss the non-monotonicity of biased games, and then in Section 3.10 we discuss new rules for Avoider-Enforcer games which force them to be bias monotone. Finally, in Section 3.11 we present several related open problems.

3.2 A sufficient condition for Avoider's win

Proof of Theorem 3.1: Our proof is based on Beck's proof of a sufficient condition for Breaker to win the $(p, q, \mathcal{H})$ Maker-Breaker game [6], which in turn is based on a method of Erdős and Selfridge [36].

Given a hypergraph $\mathcal{H}$ and disjoint subsets X and Y of the vertex set V of $\mathcal{H}$, let $\varphi(X, Y, \mathcal{H}) = \sum'_D (1 + \frac{1}{p})^{-|D \setminus X|}$ where the summation $\sum'$ is extended over those $D \in \mathcal{H}$ for which $D \cap Y = \emptyset$. Given $z \in V$, let $\varphi(X, Y, \mathcal{H}, z) = \sum''_D (1 + \frac{1}{p})^{-|D \setminus X|}$ where the summation $\sum''$ is extended over those $D \in \mathcal{H}$ for which $z \in D$ and $D \cap Y = \emptyset$.

Now, consider a play according to the rules; assume first that Avoider starts the game. Let $x_i^{(1)}, \ldots, x_i^{(p)}$ and $y_i^{(1)}, \ldots, y_i^{(q)}$ denote the vertices chosen by Avoider and Enforcer on their ith move, respectively.

Let $X_i = \{x_1^{(1)}, \ldots, x_1^{(p)}, \ldots, x_i^{(1)}, \ldots, x_i^{(p)}\}$, $Y_i = \{y_1^{(1)}, \ldots, y_1^{(q)}, \ldots, y_i^{(1)}, \ldots, y_i^{(q)}\}$, where $X_0 = \emptyset$ and $Y_0 = \emptyset$. Furthermore let $X_{i,j} = X_i \cup \{x_{i+1}^{(1)}, \ldots, x_{i+1}^{(j)}\}$ and $Y_{i,j} = Y_i \cup \{y_{i+1}^{(1)}, \ldots, y_{i+1}^{(j)}\}$ where $X_{i,0} = X_i$ and $Y_{i,0} = Y_i$. Whenever Avoider claims some vertex x, the "danger" that Avoider will completely occupy a hyperedge that contains x (and therefore lose) increases. On the other hand, if Enforcer claims some vertex y, then Avoider can never completely occupy a hyperedge that contains y, that is, such a hyperedge poses no "danger" at all for Avoider. This leads us to define the following potential function: for every non-negative integer i, let $\psi(i) = \varphi(X_i, Y_i, \mathcal{H})$. The function $\psi(i)$ describes the potential of the game after the ith round. Avoider loses iff there exists an integer i such that $D \subseteq X_i$ for some $D \in \mathcal{H}$. If this is the case then $\psi(i) \geq (1 + \frac{1}{p})^0 = 1$. It follows that if $\psi(i) < 1$ for every $i \geq 0$ then Avoider wins. Avoider's winning strategy is then the following: on his $(i+1)$st move, for every $1 \leq k \leq p$, he computes the value of $\varphi(X_{i,k-1}, Y_i, \mathcal{H}, x)$ for every vertex $x \in V \setminus (Y_i \cup X_{i,k-1})$ and then selects $x_{i+1}^{(k)}$ for which the minimum is attained. We show that the value of ψ does not increase throughout the game. If Avoider claims a vertex $x_{i+1}^{(k)}$, then the potential of every hyperedge that contains $x_{i+1}^{(k)}$ is multiplied by $1 + \frac{1}{p}$. Hence, every such hyperedge e, which currently has potential $f(e)$, adds an extra $\frac{1}{p} f(e)$ to $\psi(i+1)$. On the other hand, if Enforcer claims some vertex y, then the potential of every hyperedge that contains y drops to 0 (equivalently, this hyperedge is removed from the sum). Thus, we have

$$\psi(i+1) = \psi(i) + \frac{1}{p} \sum_{k=1}^{p} \varphi(X_{i,k-1}, Y_i, \mathcal{H}, x_{i+1}^{(k)}) - \sum_{t=1}^{q} \varphi(X_{i+1}, Y_{i,t-1}, \mathcal{H}, y_{i+1}^{(t)}). \tag{3.2}$$

Using the minimum property of $x_{i+1}^{(k)}$ and the simple observation $\varphi(X, Y, \mathcal{H}, z') \leq \varphi(X \cup \{z''\}, Y, \mathcal{H}, z')$, we get $\varphi(X_{i,k-1}, Y_i, \mathcal{H}, x_{i+1}^{(k)}) \leq \varphi(X_{i,k-1}, Y_i, \mathcal{H}, y_{i+1}^{(1)}) \leq \varphi(X_{i+1}, Y_i, \mathcal{H}, y_{i+1}^{(1)})$ for every $1 \leq k \leq p$. By (3.2) and since $\varphi(X_{i+1}, Y_{i,t-1}, \mathcal{H}, y_{i+1}^{(t)}) \geq 0$ for every $2 \leq t \leq q$, we have

$$\begin{aligned}\psi(i+1) &\leq \psi(i)+\frac{1}{p}\sum_{k=1}^{p}\varphi(X_{i,k-1},Y_i,\mathcal{H},x_{i+1}^{(k)})-\varphi(X_{i+1},Y_i,\mathcal{H},y_{i+1}^{(1)})\\ &\leq \psi(i)+\frac{1}{p}p\varphi(X_{i+1},Y_i,\mathcal{H},y_{i+1}^{(1)})-\varphi(X_{i+1},Y_i,\mathcal{H},y_{i+1}^{(1)})\\ &= \psi(i).\end{aligned}$$

Note that, on his last move, Enforcer might claim strictly less than q vertices (but at least one). This will affect the equality (3.2), but not the overall inequality $\psi(i+1) \leq \psi(i)$. Hence, the assumption $\psi(0) < 1$, entails $\psi(i) < 1$ for every i until the end of the game.

If Avoider is the last player to move then by our assumption $\psi(0) < \left(1+\frac{1}{p}\right)^{-p}$ and so $\psi(i) < 1$ for every integer i except maybe for $i = r$ which denotes the last round of the game. In this round only Avoider will participate, but then $\psi(r) \leq (1+\frac{1}{p})^p\psi(r-1) \leq (1+\frac{1}{p})^p\psi(0) < 1$.

Finally, assume that Enforcer starts the game, and on his first move he claims, say, vertices $y_1, \ldots, y_q$. Let $\tilde{\mathcal{H}}$ be the hypergraph, obtained from $\mathcal{H}$ by deleting the vertices $y_1, \ldots, y_q$ and deleting every hyperedge $e \in \mathcal{H}$ such that $e \cap \{y_1, \ldots, y_q\} \neq \emptyset$. Clearly, $\sum_{D\in\tilde{\mathcal{H}}}\left(1+\frac{1}{p}\right)^{-|D|} \leq \sum_{D\in\mathcal{H}}\left(1+\frac{1}{p}\right)^{-|D|}$. Hence, by the proof above, Avoider wins the game on $\tilde{\mathcal{H}}$ as the first player, entailing his win on $\mathcal{H}$ as the second player. This concludes the proof of the theorem. □

Proof of Theorem 3.2:
We define an infinite sequence of hypergraphs $\{\mathcal{H}_{p,q}^n\}_{n\geq 1}$ for each pair p, q of positive integers. Let $G_{p,q}^n$ be an auxiliary tree consisting of a path of length n on vertices $v_0, \ldots v_n$ with edges $e_i = (v_{i-1}, v_i)$ for every $1 \leq i \leq n$ and $p+q-1$ new leaves attached to each vertex of the path except for v_n. The set containing e_i and the $p+q-1$ edges connecting v_{i-1} to leaves, is called the *ith level.*

The vertices of $\mathcal{H}_{p,q}^n$ are the edges of $G_{p,q}^n$. The hyperedges of $\mathcal{H}_{p,q}^n$ are of the form $\{e_1, \ldots, e_{i-1}\} \cup W$, where W is a subset of the ith level, $|W| = p$ and $e_i \notin W$ or of the form $\{e_1, \ldots, e_n\} \cup W$ where W is a subset of the nth level, $|W| = p-1$ and $e_n \notin W$ (see Figure 3.1).

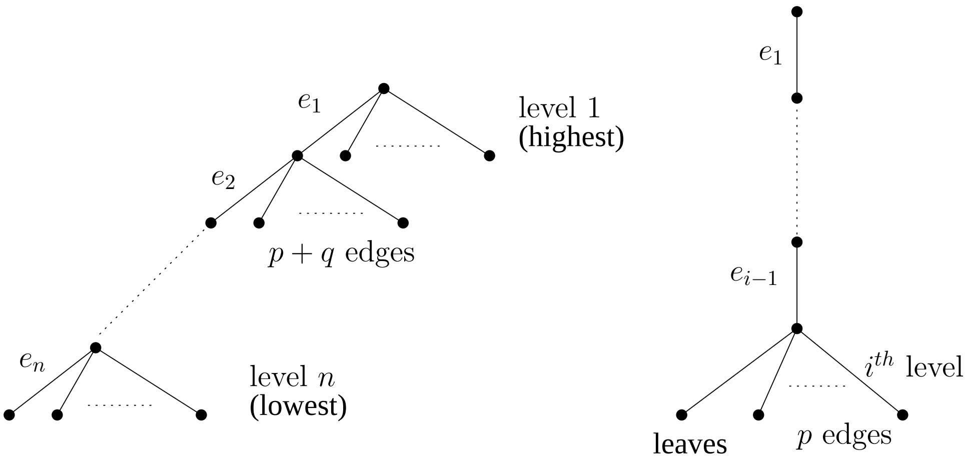

Figure 3.1: $G^n_{p,q}$ and a typical hyperedge of $\mathcal{H}^n_{p,q}$.

We have:

$$\begin{aligned}
\sum_{D\in\mathcal{H}^n_{p,q}}\left(1+\frac{1}{p}\right)^{-|D|} &= \binom{p+q-1}{p}\sum_{i=0}^{n-1}\left(1+\frac{1}{p}\right)^{-(i+p)}+\binom{p+q-1}{p-1}\left(1+\frac{1}{p}\right)^{-(n-1+p)}\\
&= \binom{p+q-1}{q-1}\left[\left(1+\frac{1}{p}\right)^{-(n-1+p)}\frac{(1+\frac{1}{p})^n-1}{1+\frac{1}{p}-1}+\frac{p}{q}\left(1+\frac{1}{p}\right)^{-(n-1+p)}\right]\\
&\le \binom{p+q-1}{q-1}p\left(1+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-p}\\
&\le \binom{p+q-1}{q-1}\frac{p+1}{2},
\end{aligned}$$

where the first inequality follows since $q \geq 1$.

Finally, we need to show that Enforcer wins the (p,q) game on $\mathcal{H}^n_{p,q}$. His strategy is very simple - he always picks edges from the lowest possible levels (level $i+1$ is considered to be lower than level i), breaking ties arbitrarily.

With this strategy he ensures that the number of edges claimed by Avoider in the first level is at least p. If this number is strictly larger than p or e_1 was claimed by Enforcer then Avoider lost. Assume then, that Avoider has claimed exactly p edges in the first level and one of them is e_1. Now, Enforcer's strategy ensures that Avoider has claimed at least p edges of the second level. If Avoider didn't claim a winning set in the second level then by the same reasoning he has claimed at least p edges of the third level and so on. Since, in the nth level, every p edges form a winning set, Avoider must have claimed one (in some level) and therefore lost. $\square$

3.3 The Avoider-Enforcer perfect matching game

Proof of Theorem 3.3

Let $0 \leq t \leq q$ be the smallest integer such that $(q+1) \mid n(n-1)+t$. Let $G=(U \cup V, E)$ be a copy of $K_{n,n}$ in K_{2n} and let F be an arbitrary set of t edges from E. Let $E_1 = E \setminus F$ and let E_2 denote the remaining edges of K_{2n}. Whenever Avoider picks an edge of E_2, Enforcer picks q edges of E_2. This is always possible as $|E_2| = n(n-1)+t$ which is divisible by $q+1$. Whenever Avoider picks an edge of E_1, Enforcer, picks q edges of E_1 (this is always possible except for maybe once). It is therefore sufficient to prove that Enforcer can win the $(1,q)$ perfect matching game on E_1.

We will provide Enforcer with a strategy, which guarantees that at the end of the game Avoider's graph will satisfy Hall's condition. To this end we define an auxiliary game which we denote by $HALL$ on E_1 with hypergraph $\mathcal{F}_{2n}$ (which is defined below), where Enforcer takes the role of Avoider (to avoid confusion, Enforcer will be referred to as "Hall-Avoider") and Hall-Avoider's win in $(q,1,\mathcal{F}_{2n})$ implies Enforcer's win in the $(1,q)$ perfect matching game.

The vertices of $\mathcal{F}_{2n}$ are the elements of E_1 and hyperedges of $\mathcal{F}_{2n}$ are all the edge-sets $E(X,Y) \subseteq E_1$ between two subsets $X \subseteq U$ and $Y \subseteq V$ for which $|X|+|Y| = n+1$. Clearly, if Hall-Avoider avoids completely occupying any such $E(X,Y)$, then in his opponent's graph $|N(X)| \geq |X|$ for every $X \subseteq U$, where $N(X) = \{v \in V : \exists u \in X, (u,v) \in E_1\}$.

We apply Theorem 3.1. For $q = cn/\log_2 n$, we have:

$$\begin{aligned}
\sum_{D \in \mathcal{F}_{2n}} \left(1+\frac{1}{q}\right)^{-|D|} \leq \sum_{D \in \mathcal{F}_{2n}} 2^{-|D|/q} &\leq \sum_{k=1}^{n} \binom{n}{k}\binom{n}{n-k+1} 2^{-\frac{(k(n-k+1)-t)\log_2 n}{cn}} \\
\leq\ & 2\sum_{k=1}^{n/2} \binom{n}{k}^2 2^{1-\frac{k(n-k+1)\log_2 n}{cn}} \\
\leq\ & 2\sum_{k=1}^{\sqrt{n}} \left[n^2 \cdot 2^{1-\frac{(n-k+1)\log_2 n}{cn}}\right]^k + 2\sum_{k=\sqrt{n}}^{n/2} \left[\left(\frac{en}{k}\right)^2 2^{1-\frac{(n-k+1)\log_2 n}{cn}}\right]^k \\
\leq\ & 2\sum_{k=1}^{\sqrt{n}} \left[2n^2 \cdot n^{-\ 1-\frac{1}{\sqrt{n}}\ \frac{1}{c}}\right]^k + 2\sum_{k=\sqrt{n}}^{n/2} \left[n \cdot 2e^2 \cdot n^{-\left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{c}}\right]^k .
\end{aligned}$$

Both sums are $o(1)$ provided $c = \frac{1}{2} - o(1)$. Hence Theorem 3.1 applies and the proof of Theorem 3.3 is complete. $\square$

Remark. Theorem 3.3 can be easily adapted to show that playing the $(1,q)$ game on K_{2n+1}, Enforcer can force Avoider's graph to admit a matching which covers all vertices but one for every $q \leq \frac{cn}{\log n}$. We omit the straightforward details.

3.4 The Avoider-Enforcer Hamilton cycle game

Proof of Theorem 3.4

We will use the following special case of Theorem 8.1 from Chapter 8 (see also [44]):

Theorem 3.9 *Let $G=(V,E)$ be a graph on n vertices and let $d=\frac{\log^{(3)} n}{\log^{(4)} n}$. Assume that G satisfies the following properties:*

P1 *For every $S \subset V$, if $|S| \le \frac{n\log^{(2)} n \log d}{d \log n \log^{(3)} n}$ then $|N(S)| \ge d|S|$.*

P2 *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \ge \frac{n\log^{(2)} n \log d}{4130 \log n \log^{(3)} n}$.*

Then G is hamiltonian, for sufficiently large n.

Let $\mathcal{H}_n^1$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are all the copies of $K_{r,r}$ in K_n where $r=\frac{n\log^{(2)} n \log d}{4130 \log n \log^{(3)} n}$. Let $\mathcal{H}_n^2$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are all the copies of $K_{s,t}$ in K_n for every $1 \le s \le \frac{n\log^{(2)} n \log d}{d \log n \log^{(3)} n}$ and $t = n - d\cdot s$. In order to win, Enforcer would like to make Avoider build a graph that satisfies the properties of Theorem 3.9, that is, Enforcer would like to avoid selecting all the edges connecting any two disjoint subsets of V of size at least $\frac{n\log^{(2)} n \log d}{4130 \log n \log^{(3)} n}$ each, and to avoid selecting all the edges connecting any two disjoint subsets of V, one of size $1 \le s \le \frac{n\log^{(2)} n \log d}{d \log n \log^{(3)} n}$ and the other of size $n - d\cdot s$. Thus, by Theorem 3.1 it suffices to prove that $\sum_{D\in\mathcal{H}_n^1\cup\mathcal{H}_n^2}(1+\frac{1}{q})^{-|D|} < (1+\frac{1}{q})^{-q}$. But

$$
\begin{aligned}
&\sum_{D\in\mathcal{H}_n^1}\left(1+\frac{1}{q}\right)^{-|D|} \le \sum_{D\in\mathcal{H}_n^1} 2^{-|D|/q} \\
\le\ & \binom{n}{\frac{n\log^{(2)} n \log d}{4130 \log n \log^{(3)} n}}^2 \exp\left\{-\log 2\left(\frac{n\log^{(2)} n\log d}{4130\log n\log^{(3)} n}\right)^2 \frac{8500\log n \log^{(3)} n}{n\log 2\log^{(4)} n}\right\} \\
\le\ & \left(\frac{4130e\log n\log^{(3)} n}{\log^{(2)} n\log d}\right)^{2\frac{n\log^{(2)} n \log d}{4130 \log n \log^{(3)} n}} \exp\left\{-\frac{(1-o(1))8500n(\log^{(2)} n)^2\log d}{(4130)^2\log n\log^{(3)} n}\right\} \\
\le\ & \exp\left\{(2+o(1))\frac{n(\log^{(2)} n)^2\log d}{4130\log n\log^{(3)} n} - \frac{8500n(\log^{(2)} n)^2\log d}{(4130)^2\log n\log^{(3)} n}\right\} \\
=\ & o(1).
\end{aligned}
$$

Similarly for every $1 \leq s \leq \frac{n \log^{(2)} n \log d}{d \log n \log^{(3)} n}$ and $D \in \mathcal{H}_n^2$ of size s we have

$$\begin{aligned}
\binom{n}{s}\binom{n}{n-d\cdot s}\left(1+\frac{1}{q}\right)^{-|D|} &\leq n^s n^{d\cdot s} 2^{-|D|/q} \\
&\leq \exp\left\{s(1+d)\log n - \log 2(n-d\cdot s)s\frac{8500 \log n \log^{(3)} n}{n \log 2 \log^{(4)} n}\right\} \\
&= o\left(\frac{1}{n}\right).
\end{aligned}$$

Thus $\sum_{D\in\mathcal{H}_n^2}(1+\frac{1}{q})^{-|D|} = o(1)$. It follows that $\sum_{D\in\mathcal{H}_n^1\cup\mathcal{H}_n^2}(1+\frac{1}{q})^{-|D|} \leq \sum_{D\in\mathcal{H}_n^1}(1+\frac{1}{q})^{-|D|} + \sum_{D\in\mathcal{H}_n^2}(1+\frac{1}{q})^{-|D|} < (1+\frac{1}{q})^{-q}$. □

3.5 Avoider-Enforcer connectivity related games

Proof of Theorem 3.8

Let $T_1, T_2, \ldots, T_{q+1}$ be pairwise edge disjoint spanning trees of $G = (V, E)$. Let $I = \bigcup_{i=1}^{q+1} T_i$ and $L = E \setminus I$. Enforcer's strategy is the following: he maintains acyclic graphs $G_1, G_2, \ldots, G_{q+1}$. In the beginning $G_i = T_i$ for every $1 \leq i \leq q+1$. Whenever Avoider picks some edge $e \in G_j$, Enforcer picks one edge $f_i \in G_i$ for every $1 \leq i \neq j \leq q+1$ (hence a total of q edges). If $G_i \cup \{e\}$ is acyclic then f_i is chosen arbitrarily. Otherwise $G_i \cup \{e\}$ contains a unique cycle C_i and then Enforcer picks some unclaimed $f_i \in C_i$. In both cases Enforcer replaces G_i with $G_i \cup \{e\} \setminus \{f_i\}$. If Avoider picks an edge of L then Enforcer picks any q previously unclaimed edges of L. If there are only $r < q$ edges left in L then Enforcer picks these r edges and another single arbitrary edge $f_i \in G_i$ for every $1 \leq i \leq q-r$. Finally, if Enforcer starts the game then on his first move he picks any q edges of L if $|L| \geq q$ and otherwise all the edges of L and one arbitrary edge $f_i \in G_i$ for every $1 \leq i \leq q-|L|$. In any case Enforcer removes f_i from G_i.

We will prove that Enforcer's strategy is a winning strategy. First, note that every unclaimed edge of I is in exactly one G_i, every edge of I claimed by Avoider is in every G_i and every edge claimed by Enforcer is in no G_i. This is clearly true in the beginning and then an edge is removed from G_i iff it is chosen by Enforcer and added to every G_i iff it is chosen by Avoider. Furthermore, after every round (a move by Avoider and a counter move by Enforcer) G_i is either a spanning tree or a spanning tree minus one edge for every $1 \leq i \leq q+1$. This is clearly true in the beginning. Assume it is still true after the kth round. If on his $(k+1)$st move Avoider picks $e \in G_j$, then Enforcer picks $f_i \in G_i$ according to his strategy. If $G_i \cup \{e\}$ is acyclic then it must be a spanning tree and so $G_i \cup \{e\} \setminus \{f_i\}$ is a spanning tree minus one edge. Otherwise $G_i \cup \{e\}$ contains a cycle C_i and since $f_i \in C_i$ (such an f_i must exist as all the G_i's were acyclic on the kth round) $G_i \cup \{e\} \setminus \{f_i\}$ is the same as G_i was (both are spanning trees or both are spanning trees minus an edge). If both players play in L then there is nothing to prove. It is possible (as was mentioned above) that there will be one (and only one) round in which Avoider does not pick any edges of I and Enforcer does. Clearly (by the above argument), before that round every G_i was a spanning tree. Now several G_i's will still be spanning trees and the rest will be spanning trees minus one edge. Thus, in the end $G_i = G_A \cap I$ for every $1 \leq i \leq q+1$, where G_A is the graph built by Avoider. It follows that $G_A \cap I$ is either a spanning tree or a spanning tree minus an edge, and since $|G_A \cap I| = |V| - 1$, the former must hold. □

Remark. The opposite implication of Theorem 3.8 is "almost" true, in the sense that it is true if we add restrictions on the number of edges and the identity of the first player. Indeed if G does not contain $q+1$ pairwise edge disjoint spanning trees then by the famous theorem of Nash-Williams [65] and independently Tutte [75], for some $r \geq 2$ there exists a partition of the vertices of G into r parts with at most $(q+1)(r-1)-1$ crossing edges. So Avoider (depending of course on who starts the game and how many edges are there in G) may claim less than $r-1$ of them and thus win.
Note that something quite different happens in the $(1, q)$ Maker-Breaker connectivity game. Here if $q \geq 2$ then the existence of $q+1$ pairwise edge disjoint spanning trees does not guarantee Maker's win. In fact there are graphs with an arbitrarily large number s of such trees which are a win for Breaker in the $(1, 2)$ game (as was already mentioned in [30]). Such a graph is for example m copies of K_{2s} such that copy i is connected by s edges to copy $i+1$ for every $1 \leq i \leq m-1$ and m is sufficiently large.
If $q = 1$ then two edge disjoint spanning trees are enough to ensure Maker's win (c.f. [60]).
If a graph G does not contains $q+1$ pairwise edge disjoint spanning trees then the outcome depends on the identity of the first player (but not necessarily on the number of edges, as Breaker wins if he starts the game). Again this follows from the theorem of Nash-Williams and Tutte.

Remark. There is a polynomial time algorithm for finding $q+1$ pairwise edge disjoint spanning trees in a graph G, in case they exist, (c.f. [34]). Thus, combined with our proof it yields an efficient explicit winning strategy for Enforcer.

Remark. The proof of Theorem 3.8 can be generalized to a game on any matroid (the matroid contains $q+1$ pairwise disjoint bases and Avoider loses iff he selects all the elements of some basis). We omit the straightforward details.

Proof of Theorem 3.5
If $q \leq \lfloor \frac{n}{2} \rfloor - 1$ then Enforcer wins the game by Theorem 3.8 as K_n contains $\lfloor \frac{n}{2} \rfloor$ pairwise edge disjoint spanning trees. If Avoider is the first player in the $(1, \lfloor \frac{n}{2} \rfloor)$ game on K_n, where n is odd, then at the end of the game he will have exactly $n-1$ edges. So he will win iff he will claim all the edges of some cycle in K_n. This Maker-Breaker game where Maker's goal is to build a cycle, was studied by Bednarska and Pikhurko [22] in a more general context. In the particular case that we are interested in, their result shows that Breaker can break all the cycles and so Enforcer can force Avoider to build a spanning tree. Finally, in any other case, Avoider will not have enough edges to build a spanning tree so he will win no matter how he plays. □

Proof of Theorem 3.6
If $q \geq \frac{n}{k}$ then at the end of the game Avoider will have at most $\lceil \frac{n(n-1)}{2(\frac{n}{k}+1)} \rceil \leq \frac{n(n-1)}{2(\frac{n}{k}+1)} + 1$ edges. Thus the minimum degree in Avoider's graph, regardless of his strategy, will be at most $\frac{n-1}{\frac{n}{k}+1} + \frac{2}{n} < k$ where the last inequality holds for every $k \geq 2$. It follows that Avoider's graph will not be k-edge connected.

Let $q \leq \frac{n}{2k} - 1$ and let $T_1, \ldots, T_{\lfloor n/2 \rfloor}$ be pairwise edge disjoint spanning trees of K_n. For every $1 \leq i \leq k$, let $G_i = \bigcup_{j=(i-1)\lfloor \frac{n}{2k} \rfloor + 1}^{i \lfloor \frac{n}{2k} \rfloor} T_j$. Enforcer plays k separate games in parallel, that is, whenever Avoider claims an edge of G_i, for some $1 \leq i \leq k$, Enforcer plays all his q edges in G_i. By Theorem 3.8, Enforcer can make Avoider build a spanning tree in G_i for every $1 \leq i \leq k$ (as in every G_i there are $\lfloor \frac{n}{2k} \rfloor$ pairwise edge disjoint spanning trees and $q \leq \lfloor \frac{n}{2k} \rfloor - 1$). The G_i's are pairwise edge disjoint and so Avoider's graph will be k-edge connected. □

Proof of Theorem 3.7
Similarly to the second part of the proof of Theorem 3.6, Enforcer plays $\lfloor n/4 \rfloor$ separate games in parallel. The board of each of these games consists of two edge disjoint spanning trees, and so, by Theorem 3.8, Enforcer can make Avoider build one spanning tree on every board and hence a total of $\lfloor n/4 \rfloor$ trees. □

3.6 The Avoider-Enforcer planarity game

In the following theorem we give an upper bound and a lower bound for the threshold bias at which Enforcer's win turns into an Avoider's win in the planarity game.

Theorem 3.10
$$\frac{n}{2} - o(n) \leq f^-_{\mathcal{NP}_n} \leq f^+_{\mathcal{NP}_n} \leq 2n^{5/4}.$$

Proof: Assume first that $b \geq 2n^{5/4}$. We will provide Avoider with a strategy for building a planar graph. The game is divided into four stages. Avoider's strategy is the following.

In the first stage, Avoider builds a matching by repeatedly claiming an edge that connects two vertices, neither of which is incident with any other edge previously claimed by him. The first stage ends when no such unclaimed edge remains and so Avoider cannot further extend his matching. We denote the set of vertices that are covered by Avoider's matching by M.

Next, in the second stage Avoider claims edges with one endpoint in M and the other in $V \setminus M$ such that throughout the second stage every vertex of $V \setminus M$ has degree at most one in Avoider's graph. The second stage ends when no such unclaimed edge remains.

In the third stage Avoider builds another matching on M. More precisely, he repeatedly claims edges that connect two vertices of M, neither of which is incident with any other edge previously claimed by him in the **third** stage. The third stage ends when no such unclaimed edge remains and Avoider cannot further extend this second matching.

In the fourth and final stage, Avoider claims edges arbitrarily to the end of the game. If we prove that in this stage Avoider will claim at most one edge, then the upper bound of the theorem will follow. Indeed, the graph that is spanned by Avoider's edges from the first and third stages is a union of two matchings, that is, a union of disjoint paths and cycles. Furthermore, if we add Avoider's edges from the second stage to this graph, then we simply add "hanging" edges (edges with one endpoint having degree one). Clearly, if we now add any single edge that may have been claimed by Avoider in the fourth stage to that graph, it remains planar.

Let e be the number of edges that Avoider claims in the entire game. By the end of the first stage, Enforcer must have claimed all the edges with both endpoints in $V \setminus M$. Since Avoider's matching on M consists of at most e edges, we have $|V \setminus M| \geq n - 2e$ and therefore Enforcer has already claimed at least $\binom{n-2e}{2}$ edges. It follows that there are at most

$$\binom{n}{2} - \binom{n-2e}{2} \leq 2en$$

unclaimed edges left in the graph and Avoider will claim at most $\frac{2en}{b}$ of them.

In the second stage, Avoider claims edges between M and $V \setminus M$. When this is no longer possible, every unclaimed edge between M and $V \setminus M$ is incident with a vertex of $V \setminus M$ which has degree one in Avoider's graph. It follows that, at this point, the number of unclaimed edges between M and $V \setminus M$ is at most

$$2e \cdot \frac{2en}{b} = \frac{4e^2n}{b}.$$

In the third stage, Avoider builds his second matching on M. When this is no longer possible, the number of unclaimed edges with both endpoints in M is at most

$$\binom{2e}{2} - \binom{2e - 4en/b}{2} \leq \frac{8e^2n}{b}.$$

To see this, it is enough to observe that the number of vertices that are incident with the second matching is at most $4en/b$, and that all edges with endpoints in M that are not adjacent to the second matching must be claimed by Enforcer after the third stage.

Putting everything together, the total number of unclaimed edges after the third stage is at most

$$\frac{4e^2n}{b} + \frac{8e^2n}{b} = \frac{12e^2n}{b}.$$

Since $e < n^2/2b$, we have that the number of edges to be played in the fourth stage is at most $\frac{12e^2n}{b} \leq \frac{3n^5}{b^3} \leq b$, which means that in the fourth stage Avoider will claim at most one edge.

Next, fix an $\varepsilon > 0$ and assume that $b \leq \frac{n}{2}(1-\varepsilon)$. We will provide Enforcer with a strategy, which guarantees that Avoider will occupy the edges of a non-planar graph. If $b \leq n/7$, then the number of edges Avoider claims in the entire game is at least $\lfloor \binom{n}{2} \cdot (b+1)^{-1} \rfloor > 3n$, and thus Avoider surely loses regardless of Enforcer's strategy. Hence, from now on we can assume that $b > n/7$. Let $k = k(\varepsilon)$ be the smallest positive integer such that $\frac{1}{1-\varepsilon/2} > \frac{k}{k-2}$. Enforcer's strategy will be to prevent Avoider from claiming a cycle of length smaller than k, which we will call a "short cycle". If he succeeds, then at the end of the game Avoider's graph will have at least

$$\left\lfloor \frac{\binom{n}{2}}{b+1} \right\rfloor \geq \frac{n}{1-\varepsilon/2} > \frac{k}{k-2} n$$

edges for sufficiently large n, and girth at least k. As we have mentioned before, a graph with such properties cannot be planar, thus Enforcer wins.

It remains to show that Enforcer can indeed prevent Avoider from claiming a short cycle. In order to do that we will use the following theorem of Bednarska and Łuczak.

Theorem 3.11 *[20, Theorem 1] For every graph G which contains at least three non-isolated vertices there exist positive constants c and n_0 such that, playing the $(1, q)$ game on K_n, G-Breaker can prevent G-Maker from building a copy of G provided that $n > n_0$ and $q > cn^{1/m_2(G)}$, where*

$$m_2(G) = \max_{\substack{H \subseteq G \\ v(H) \geq 3}} \frac{e(H) - 1}{v(H) - 2}.$$

For a cycle C_i of length i, we have $m_2(C_i) = \frac{i-1}{i-2}$. Therefore, there exist constants c_i, $i = 3, \ldots, k-1$ such that for sufficiently large n, Enforcer can prevent Avoider from claiming a copy of C_i, if the number of edges he is allowed to claim per move is at least $c_i n^{\frac{i-2}{i-1}}$. Since for sufficiently large n

$$\sum_{i=3}^{k-1} c_i n^{\frac{i-2}{i-1}} \leq \frac{n}{7} \leq b,$$

Enforcer can simultaneously prevent Avoider from claiming any short cycle C_i, $3 \leq i < k$, by simply playing all $k-3$ games in parallel. That is, after Avoider claims an edge, Enforcer responds by claiming $c_3 n^{\frac{1}{2}}$ edges according to the strategy in the "triangle avoidance game", then he claims $c_4 n^{\frac{2}{3}}$ edges according to the strategy in the "4-cycle avoidance game", and so on. His different strategies, for the different cycle-games, might call for claiming the same edge more than once, in which case he just claims an arbitrary unclaimed edge instead. It is easy to see that this cannot harm him. This concludes the proof of the theorem. $\square$

3.7 The Avoider-Enforcer k-colorability game

In the following theorem we give an upper bound and a lower bound for the threshold bias at which Enforcer's win turns into an Avoider's win in the k-colorability game.

Theorem 3.12 *For every $k \geq 3$ there exists a constant s'_k, such that*

$$s'_k n \leq f^-_{\mathcal{NC}^k_n} \leq f^+_{\mathcal{NC}^k_n} \leq 2kn^{1+\frac{1}{2k-3}}.$$

Moreover, $s'_k \sim \frac{\log 2}{2k \log k}$ as $k \to \infty$.

Proof: Assume first that $b \leq \frac{n}{ck \log k}$. We will provide Enforcer with a strategy which ensures that by the end of the game, Avoider's graph will not be k-colorable. Enforcer's goal will be to avoid building a clique of size $\lceil n/k \rceil$. If he achieves this goal, Avoider's graph will not contain an independent set of size $\lceil n/k \rceil$ and so will not be k-colorable; thus Enforcer will win. Let $\mathcal{F}$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are the $\lceil n/k \rceil$-cliques of K_n. We name the players of the $(b, 1, \mathcal{F})$ game CliqueAvoider and CliqueEnforcer. As mentioned above, Enforcer will win the k-colorability game if he will not claim all vertices in any hyperedge of $\mathcal{F}$, that is, if he is able to win as CliqueAvoider.

We have

$$\begin{aligned}\sum_{D \in \mathcal{F}} \left(1+\frac{1}{b}\right)^{-|D|} &\leq \binom{n}{\lceil n/k \rceil}\left(1+\frac{1}{b}\right)^{-\binom{\lceil n/k \rceil}{2}} \leq (ek)^{\lceil n/k \rceil} 2^{-\binom{\lceil n/k \rceil}{2}/b} \\ &\leq 2^{\frac{n \log_2 e}{k} + \frac{n \log_2 k}{k} + \log_2(ek) - \frac{cn^2 k \log k}{2k^2 n} + \frac{nck \log k}{2kn}} = o(1).\end{aligned}$$

Applying Theorem 3.1 we conclude that there exists a winning strategy for CliqueAvoider, and thus Enforcer wins the k-colorability game.

Next, let $b > 2kn^{1+\frac{1}{2k-3}}$. We will provide Avoider with a strategy for building a $(k-1)$-degenerate graph (a graph G is called r-degenerate if there is an ordering of the vertices, $v_1, \ldots, v_n$, such that every vertex has at most r neighbors with a higher index). Clearly, that would entail Avoider's win in the k-colorability game as every $(k-1)$-degenerate graph is k-colorable.

Avoider will play several auxiliary minigames one after the other, never starting a new minigame before finishing the previous one, until all edges are claimed and the k-colorability game is over. Before we describe his strategy in detail, let us define two basic types of *minigames*.

Minigame Type I. For a set of vertices A, the (A)-*minigame* is played on those edges with both endpoints in A which are still unclaimed at the beginning of this minigame. Note that some edges within A may have already been claimed during previous minigames. We say that the vertices of A are *designated* to the (A)-minigame. When we say that *Avoider is playing the (A)-minigame*, we mean that Avoider is repeatedly claiming independent edges with both endpoints in A for as long as possible, that is, he extends a matching on A until it is no longer possible. When Avoider cannot further extend his matching, the (A)-minigame is over. At this point we denote the set of vertices of A incident with an edge, claimed by Avoider in **this** minigame by A_1, and let $A_2 = A \setminus A_1$. Note that by the end of the (A)-minigame, all edges with both endpoints in A_2 have already been claimed by one of the players.

Minigame Type II. Let A and B be two disjoint sets of vertices. The $(A:B)$*-minigame* is played on those edges with one endpoint in A and the other in B which are still unclaimed at the beginning of this minigame. Again, we assume that the (big) game is in progress, meaning that some of the edges between A and B may have already been claimed in previous minigames. We say that the vertices of B are *designated* to the $(A:B)$-minigame. When we say that *Avoider is playing the* $(A:B)$*-minigame*, we mean that Avoider is repeatedly claiming edges between A and B such that no vertex in B is incident with more than one of Avoider's edges claimed in **this** minigame. When this is no longer possible, the $(A:B)$-minigame is over. At this point, let B_1 denote the set of vertices of B that are incident with an edge claimed by Avoider in **this** minigame, and let $B_2 = B \setminus B_1$. Note that all edges with one endpoint in A and the other in B_2 have already been claimed by one of the players. The vertices in B_2 are called *finished*.

Now we can describe the way Avoider plays the game. We introduce *a minigame pool* $\mathcal{P}$, which is a dynamic collection of minigames that will be updated during the game – it will contain minigames waiting to be played by Avoider. At each moment $\mathcal{P}$ will contain at most one minigame of Type I and at most $k-1$ minigames of Type II.

Avoider will maintain a partial ordering of the vertices, which he will refine whenever a minigame is over. In this partial ordering, the vertices designated to the same minigame will be incomparable to each other, the vertices designated to the lone minigame of Type I in the pool will be above all the other vertices and for any minigame $(A:B)$ of Type II, every vertex of A will be above every vertex of B.

Given a partial ordering, let the *up-degree* of v be the number of Avoider's edges (v,u) where either u is above v or they are incomparable.

To each minigame in the pool, we assign an integer parameter, that will help us keep track of the degeneracy of Avoider's graph throughout the game. Thus, instead of the (A)-minigame (or the $(A:B)$-minigame), we will consider the $(A)_l$-minigame (or the $(A:B)_l$-minigame) for an appropriate integer l. During play, Avoider maintains the following property: if a vertex is designated to a minigame with parameter l, then its up-degree is at most l.

In the beginning of the game $\mathcal{P}$ contains only one minigame – the $(V(K_n))_0$-minigame. The partially ordered set on $V(K_n)$ contains no relations.

We say that the size of an (A)-minigame is $\frac{1}{2}|A|^2$, and the size of an $(A:B)$-minigame is $|A|\cdot|B|$. Note that the size of a minigame is an upper bound on the number of edges it contains.

When the game is played, Avoider repeatedly chooses a minigame of the largest size in the pool $\mathcal{P}$, removes it from the pool, plays it to its end, and then updates $\mathcal{P}$ and the partial ordering as follows. If the minigame played was an $(A)_l$-minigame, then he places two new minigames into $\mathcal{P}$, the $(A_1)_{l+1}$-minigame and the $(A_1:A_2)_l$-minigame. Furthermore, the designation of the vertices of A is lifted and replaced by that of the vertices of A_1 to the $(A_1)_{l+1}$-minigame and that of the vertices of A_2 to the $(A_1:A_2)_l$-minigame. The partial order is refined by placing every vertex of A_1 above every vertex of A_2.

On the other hand, if the minigame played was an $(A : B)_l$-minigame, then Avoider places only the $(A : B_1)_{l+1}$-minigame back into $\mathcal{P}$. Furthermore, the designation of the vertices of B is lifted – the vertices of B_2 are finished and the vertices of B_1 are designated to the $(A : B_1)_{l+1}$-minigame. The partial order is not affected.

This shows that indeed in every stage of the game $\mathcal{P}$ will contain at most (in fact, exactly) one minigame of Type I.

Note that at any point of the game, every unclaimed edge is in exactly one of the minigames in $\mathcal{P}$. Moreover, every vertex of K_n will be either finished or designated to exactly one minigame in the pool.

Note that after having played an $(A : B)$-minigame of Type II, the up-degree of the finished vertices, that is, the vertices of B_2, is fixed and in particular will not be increased in later stages of the game. This is because there are simply no more unclaimed edges which go to higher or incomparable vertices left. Indeed, the edges with both endpoints in B were all claimed during that minigame of Type I after which the vertices of B were designated to a Type II minigame. The edges (u, v), where $u \in B_2$ and v is above u were all claimed during the $(A : B)$-minigame. Furthermore, the up-degree of every vertex of B_2 was not changed during the $(A : B)$-minigame, so if the parameter of the $(A : B)$-minigame was l, then the up-degree of the vertices in B_2 is at most l at the end of the game.

It is clear that as long as Avoider follows this strategy and the parameter of every minigame in $\mathcal{P}$ is at most l, Avoider's graph is l-degenerate. Therefore, it suffices to prove that after the first minigame with parameter $k - 2$ is taken out of the pool to be played, Avoider plays at most one more move in the whole game. Note that whenever a minigame of Type II is played the size of the pool $\mathcal{P}$ is not changed, and whenever a minigame of Type I is played both the size of the pool $\mathcal{P}$ and the parameter of the new minigame of Type I are increased by one. It follows that proving the above will show that indeed, throughout the game, there will be at most $k - 1$ minigames of Type II in $\mathcal{P}$.

We will prove by induction on l that any minigame in the pool which has parameter $0 \le l \le k-2$ is of size at most $n^2 \left(\frac{2k^2n^2}{b^2}\right)^l$. First, for the base step, note that the size of any minigame with parameter $l = 0$ is less than n^2. Now let us assume that l is an integer with $0 < l \le k - 2$ and the induction hypotheses holds for all games with parameter less than l. For a minigame M in the pool with parameter l we consider three cases.

Case 1. M is an $(A_1)_l$-minigame that was inserted into the pool after the $(A)_{l-1}$-minigame has ended. Just before Avoider started playing the $(A)_{l-1}$-minigame there was no minigame in the pool of larger size. Since the total number of games in the pool was at most k, the total number of unplayed edges at that point was at most k times the size of the $(A)_{l-1}$-minigame. By the induction hypotheses, this is at most $kn^2 \left(\frac{2k^2n^2}{b^2}\right)^{l-1}$. The number of edges Avoider will play during the $(A)_{l-1}$-minigame is certainly bounded from above by the total number of edges that Avoider will claim until the end of the whole k-colorability game, which is at most $\frac{kn^2}{b} \left(\frac{2k^2n^2}{b^2}\right)^{l-1}$. Avoider's strategy for the $(A)_{l-1}$-minigame guarantees that the set A_1

will be of size at most twice this much, and hence the $(A_1)_l$-minigame will be of size at most

$$\frac{1}{2}|A_1|^2 \leq \frac{1}{2}\left(\frac{2kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{l-1}\right)^2 \leq n^2 \cdot \left(\frac{2k^2 \cdot n^2}{b^2}\right)^l .$$

Case 2. M is an $(A_1 : A_2)_l$-minigame that was inserted into the pool after the $(A_1 \cup A_2)_l$-minigame has ended. The size of the $(A_1 : A_2)_l$-minigame is obviously bounded from above by the size of the $(A_1 \cup A_2)_l$-minigame, which we already upper-bounded in Case 1.

Case 3. M is an $(A : B_1)_l$-minigame that was inserted into the pool after the $(A : B)_{l-1}$-minigame has ended. As in Case 1, we can bound the number of edges Avoider will play during the $(A : B)_{l-1}$-minigame from above, by the total number of edges that Avoider will claim until the end of the whole k-colorability game. Thus, knowing that the $(A : B)_{l-1}$-minigame was of maximal size in $\mathcal{P}$ before it was played, we get that Avoider will make at most

$$\frac{k|A||B|}{b} \leq \frac{kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{l-1}$$

moves until the end of the game. Therefore, the size of B_1 is also at most that much. Since the size of A is at most n^2/b (the total number of vertices that can have positive degree in Avoider's graph), the total size of the $(A : B_1)_l$-minigame is at most $n^2\left(\frac{2k^2n^2}{b^2}\right)^l$. This concludes the induction step.

At the point when a minigame with parameter $k-2$ becomes the largest size in the pool, then the total number of edges to be played in the remainder of the game is at most $kn^2\left(\frac{2k^2n^2}{b^2}\right)^{k-2}$ which is less than b, meaning that Avoider will play at most one move before the game ends. However, at this point Avoider's graph is $(k-2)$-degenerate, so we are done. $\square$

Remark: The graph built by Avoider in the proof of Theorem 3.10 is clearly 3-colorable. It follows that if $k \geq 3$ and $b \geq 2n^{5/4}$ then Avoider can win the $(1, b)$ k-colorability game. For $k = 3$ this yields a better result than the one given in Theorem 3.12. Moreover, It is easy to see that if $b > n^{3/2}$, then Avoider can build a graph which consists of a matching and one additional edge; clearly such a graph is 2-colorable. Hence, using Enforcer's strategy from Theorem 3.12 we get

$$cn/2 \leq f^{-}_{\mathcal{NC}^2_n} \leq f^{+}_{\mathcal{NC}^2_n} \leq n^{3/2},$$

for an appropriate constant $c > 0$.

3.8 The Avoider-Enforcer minor game

In the following theorem we give an upper bound and a lower bound for the threshold bias at which Enforcer's win turns into an Avoider's win in the K_t-minor game for every fixed $t \geq 4$.

Theorem 3.13 *For every $t \geq 4$,*

$$\frac{n}{2} - o_t(n) \leq f^-_{\mathcal{M}^t_n} \leq f^+_{\mathcal{M}^t_n} \leq 2n^{5/4}.$$

Before proving this theorem, we will state and prove a graph-theoretic lemma, which may also be of independent interest.

Lemma 3.14 *Let $G = (V, E)$ be a graph on n vertices, with average degree $2 + \alpha$ for some $\alpha > 0$ and girth $g^* \geq \left(1 + \frac{2}{\alpha}\right)(4 \log_2 t + 2 \log_2 \log_2 t + c)$ where c is an appropriate constant. Then G admits a K_t-minor.*

Proof: In the proof of the lemma we will use the following result of Kühn and Osthus (a similar result was also obtained by Diestel and Rempel [32]).

Theorem 3.15 *[59, Corollary 5] Let $t \geq 3$ be an integer. There exists a constant c such that every graph of minimum degree at least 3 and girth at least $4 \log_2 t + 2 \log_2 \log_2 t + c$ contains a K_t-minor.*

We repeatedly apply two deletion operations on G, which do not decrease the average degree and (trivially) do not decrease the girth. The first operation is the deletion of a vertex of degree at most one. Such an operation obviously does not decrease the average degree. In the second type of operation, given a path $u_1, u_2, \ldots, u_k$, with $k \geq 2 + 2/\alpha$, such that each of the internal vertices $u_2, u_3, \ldots, u_{k-1}$ has degree two in G, we remove $u_2, \ldots, u_{k-1}$. Again the average degree of the new graph is at least $2 + \alpha$. To verify this, let us assume that from a graph with e edges and v vertices satisfying $\frac{2e}{v} \geq 2 + \alpha$ we remove the internal vertices of a path with $k \geq 2 + 2/\alpha$ vertices. Then we obtain a graph with average degree at least

$$\frac{2(e - k + 1)}{v - (k - 2)} \geq 2 + \alpha,$$

as claimed. Let G_2 be the graph we obtain from G by repeated applications of these two operations. Since the average degree was not decreased in any step of the process, G_2 is not empty. It also follows that $\delta(G_2) \geq 2$, the girth of G_2 is at least g^* and every path of G_2, with internal vertices of degree two, is of length at most $1 + 2/\alpha$. Let G_3 denote the graph obtained from G_2 by contracting every path $u_1, u_2, \ldots, u_k$, such that u_i has degree two in G_2 for every $1 < i < k$, into a single edge. Again, this operation does not decrease the average degree and therefore G_3 is not empty. Clearly $\delta(G_3) \geq 3$. Moreover, since every such path in G_2 is of length at most $1 + 2/\alpha$ it follows that the girth of G_3 is at least $g = \frac{g^*}{1+2/\alpha} \geq 4 \log_2 t + 2 \log_2 \log_2 t + c$. Applying Theorem 3.15 we conclude that G_3 admits a K_t-minor. Since G_3 was obtained from G by the deletion and contraction of edges (and the removal of isolated vertices), G admits the same minor and the proof is complete. $\Box$

Proof of Theorem 3.13 Assume that $b \leq (1/2 - \varepsilon)n$. Let c' be the constant from Theorem 2.9. If $b < n/(2c't\sqrt{\log t})$, then the average degree of Avoider's graph at the end of the game is at least $c't\sqrt{\log t}$, and so he had lost by Theorem 2.9. Otherwise, let $\alpha = \alpha(n, \varepsilon) > 0$ be the real number that satisfies

$$(1+\alpha)n = \frac{\binom{n}{2}}{(1/2-\varepsilon)n+1}.$$

Let G_A denote Avoider's graph after he has claimed exactly $(1+\alpha/2)n$ edges. Note that the average degree in G_A is $2+\alpha$. We will prove that Avoider has lost already at this point. Playing as in the proof of Theorem 3.10, Enforcer can make sure that G_A will have girth $g^* \geq \left(1+\frac{2}{\alpha}\right)(4\log_2 t + 2\log_2\log_2 t + c)$ where c is the constant given by Theorem 3.15 (note that the ways we choose the required girth here and in the proof of Theorem 3.10 are different; however, the only thing that matters is that the girth is a constant depending only on ε and t). By Lemma 3.14, G_A admits a K_t-minor.

If $b \geq 2n^{5/4}$ then playing as in the proof of Theorem 3.10, Avoider builds a graph that does not admit a K_4-minor. $\square$

Remark: As in the remark following the proof of Theorem 3.12, if $b > n^{3/2}$, then Avoider can create a K_3 minor free graph. Moreover, our strategy for Enforcer is valid in this case.

Assume that, as in the Maker-Breaker case, Enforcer would like to make Avoider build a graph that admits a K_t-minor for a non-constant t, that is, for t which tends to infinity with n. We do not know if he can achieve this with bias $(1-\varepsilon)n/2$, but we can prove that he can win with a, smaller yet still linear, bias.

Theorem 3.16 *If $b \leq n/19$, then Enforcer has a winning strategy for the $(1,b)$ game $\mathcal{M}_n^t$, for every $t < c\sqrt{n/\log n}$, where c is some absolute constant.*

In the proof of Theorem 3.16, we will use a result of Plotkin, Rao and Smith. Before we can state their result, we need the following definition:

Definition Let $G = (V, E)$ be a graph on n vertices. A set $S \subset V$ is called a *separator* if every connected component of $G[V \setminus S]$ contains at most $2n/3$ vertices.

Theorem 3.17 *[68, Corollary 2.4] Let G be a graph on n vertices and let h be a function of n. If G does not have a separator of size at most $O(h\sqrt{n\log n})$, then G admits a K_h minor.*

Proof of Theorem 3.16 Assume that $b \leq n/19$; we will present a winning strategy for Enforcer. We will prove that Enforcer can make Avoider build a graph in which there is an edge between any two disjoint vertex sets of size at least $s = (1/3-\varepsilon)n$ each, where $\varepsilon > 0$ is some small constant. Let $\mathcal{H}_n$ denote the hypergraph whose vertices are the edges of K_n and whose hyperedges are all the subgraphs of K_n, isomorphic to $K_{s,s}$. Enforcer's goal is to avoid claiming any hyperedge of $\mathcal{H}_n$. Using the criterion given by Theorem 3.1, we obtain

$$\sum_{D\in\mathcal{H}_n}\left(1+\frac{1}{b}\right)^{-|D|} \leq \binom{n}{n/3}\binom{n}{n/3}2^{-n^2/(9+\varepsilon')b} = o(1).$$

Thus, Enforcer can make sure that Avoider's graph will not contain a separator of size at most εn and so, by Theorem 3.17, Avoider's graph will admit a K_h minor for $h = \varepsilon\sqrt{n/\log n}$. This concludes the proof of the Theorem. □

3.9 Non-monotonicity of biased games

In this section we give two examples which show that Avoider-Enforcer games are not monotone in general. It would be extremely interesting to give at least a sufficient condition for an Avoider-Enforcer game to be monotone.

Though quite straightforward, for the sake of completeness, we prove that Maker-Breaker games are indeed monotone.

Proposition 3.18 *If Maker wins the $(p, q, \mathcal{H})$ game, where $\mathcal{H}$ is any hypergraph, then he also wins the $(p+1, q, \mathcal{H})$ game and the $(p, q-1, \mathcal{H})$ game. Similarly, If Breaker wins the $(p, q, \mathcal{H})$ game, then he also wins the $(p-1, q, \mathcal{H})$ game and the $(p, q+1, \mathcal{H})$ game.*

Proof: Assume first that Maker has a winning strategy $\mathcal{S}_m$ for the $(p, q, \mathcal{H})$ game. When playing the $(p, q-1, \mathcal{H})$ game, Maker plays according to $\mathcal{S}_m$. Whenever Breaker picks his $q-1$ vertices, Maker (in his mind) chooses an arbitrary unclaimed vertex and 'gives' it to Breaker. If Breaker picks an unclaimed vertex that already 'belongs' to him in Maker's mind, then Maker 'gives' him another arbitrary unclaimed vertex. By the end (in Maker's mind) of the game he has already won (as he played according to $\mathcal{S}_m$). Clearly, no matter how they proceed, Maker will win the game.

When playing the $(p+1, q, \mathcal{H})$ game, Maker plays according to $\mathcal{S}_m$, where in every turn he picks one additional arbitrary unclaimed vertex. At a certain point during the game it might happen that a vertex that Maker should pick according to $\mathcal{S}_m$ already belongs to him; he will then pick another arbitrary unclaimed vertex. Since Maker played according to $\mathcal{S}_m$, at the end of the game, if we remove all the 'additional' vertices picked by Maker, we get a position from which Maker can win. Clearly, picking every remaining vertex is a winning strategy.

The proof of monotonicity for Breaker's win is analogous. □

Next, we give examples showing that Avoider-Enforcer games need not be monotone.

Example 1: Consider the $(1, q, \mathcal{H}_t)$ game, where the vertices of $\mathcal{H}_t$ are the vertices of $t \cdot K_2$ (i.e. t vertex disjoint edges), and the hyperedges of $\mathcal{H}_t$ are the edges of $t \cdot K_2$. We claim that, for sufficiently large t, Enforcer (as first or second player) wins this game iff q is even. Indeed, if q is even then in every turn (for as long as possible) Enforcer picks $\frac{q}{2}$ edges (unclaimed pairs of vertices). Clearly, if $t > q(\frac{q}{2}+1)$ then Avoider will lose. If q is odd, then assuming that Enforcer is the first player, Avoider can always pick an unclaimed

vertex whose single neighbor was picked by Enforcer, and therefore win. Finally, if Avoider is the first player, then in every turn, either Enforcer picks all unclaimed vertices which are in the neighborhood of Avoider's vertices, or he picks a vertex whose single neighbor w is unclaimed. In the former case, Avoider claims an arbitrary vertex, whereas in the latter, he claims w. Either way, after every move of Avoider, there is at most one unclaimed vertex in the neighborhood of Avoider's vertices.
It follows that this game in not monotone in q.

Example 2: Consider the $(p, 1, \mathcal{H}'_t)$ game, where the vertices of $\mathcal{H}'_t$ are the vertices of $t \cdot K_2$ and the hyperedges of $\mathcal{H}'_t$ are the minimal vertex covers of $t \cdot K_2$. We claim that, for sufficiently large t, Avoider (as first or second player) wins this game iff p is even. This follows immediately from the analysis of **Example 1** if Avoider (Enforcer) adopts Enforcer's (Avoider's) strategy from that example.

It follows that this game in not monotone in p.

Note that though in general Avoider-Enforcer games are not monotone, the sufficient condition given in Theorem 3.1 guarantees monotonicity in both p and q. Indeed, $\left(1 + \frac{1}{p}\right)^{-1}$ is monotone increasing in p and so if $\sum_{D \in \mathcal{H}} \left(1 + \frac{1}{p}\right)^{-|D|} < e^{-1}$ then $\sum_{D \in \mathcal{H}} \left(1 + \frac{1}{r}\right)^{-|D|} < e^{-1}$ for every $r \leq p$. Hence, it follows from Theorem 3.1 that if $\sum_{D \in \mathcal{H}} \left(1 + \frac{1}{p}\right)^{-|D|} < e^{-1}$, then Avoider wins the (r, q) game for every $r \leq p$ and $q \geq 1$.

3.10 Monotone rules for Avoider-Enforcer games

In order to overcome the non-monotonicity of Avoider-Enforcer games and, as a result, the lack of a well defined threshold bias, we offer a change in the rules of Avoider-Enforcer games (this was suggested independently by Malgorzata Bednarska, Tomasz Ł uczak and Nati Linial). The rules stay the same as in the usual case (throughout this subsection we refer to these rules as the *strict rules*) except that both players are allowed to claim more elements per turn. That is, in a $(p, q, \mathcal{H})$ game, Avoider claims **at least** p elements of $\mathcal{H}$ per turn (instead of **exactly** p) and Enforcer claims **at least** q elements of $\mathcal{H}$ per turn. It is easy to see that Avoider-Enforcer games with these rules are monotone and thus have a threshold bias. We denote this threshold by $f_m(\mathcal{H})$ and refer to these rules as the *monotone rules.*

It seems reasonable to expect that an Avoider-Enforcer game $\mathcal{H}$ played with the strict rules or with the monotone rules will have a similar outcome; more accurately it is plausible that

$$f^-_{\mathcal{H}} \leq f_m(\mathcal{H}) \leq f^+_{\mathcal{H}} \tag{3.3}$$

for any Avoider-Enforcer game $\mathcal{H}$. Note that if we change the rules of Maker-Breaker games analogously, that is we allow both players to claim per turn a number of elements of the board which is **at most** their bias, then not only the outcome remains the same, but even the optimal strategies themselves do not change. Moreover, a quick look at its proof, shows that the sufficient condition for Avoider's win in the $(p, q, \mathcal{H})$ strict game, which given by Theorem 3.1, holds for the $(p, q, \mathcal{H})$ monotone game as well:

Let $\mathcal{D}_n$ denote the hypergraph whose vertices are the edges of K_n and whose hyperedges are the edge sets of all spanning subgraphs of K_n of positive minimum degree. The main result of this subsection is the following theorem:

Theorem 3.19 *For sufficiently large n, Avoider has a winning strategy for the $(1, \frac{n-1}{\log(n-2)})$ minimum positive degree monotone game, played on the edges of K_n; that is, $f_m(\mathcal{D}_n) \leq \frac{n-1}{\log(n-2)}$.*

From Theorem 3.19 and the monotone rules version of Theorem 3.1 we immediately obtain the following interesting corollaries:

Corollary 3.20 *Let $\mathcal{D}_n^k$, $\mathcal{T}_n^k$, $\mathcal{M}_n$, $\mathcal{H}_n$, denote the hypergraphs whose vertices are the edges of K_n and whose hyperedges are the edge sets of all subgraphs of K_n with minimum degree k, the edge sets of all k-connected spanning subgraphs of K_n, the edge sets of all matchings of K_n that cover at least $n-1$ of its vertices and the edge sets of all Hamilton cycles of K_n respectively; then*

$$(1 - o(1)) \frac{n}{\log n} \leq f_m(\mathcal{D}_n^k), f_m(\mathcal{T}_n^k), f_m(\mathcal{M}_n), f_m(\mathcal{H}_n) \leq (1 + o(1)) \frac{n}{\log n}. \tag{3.4}$$

Indeed, the leftmost inequality in (3.4) follows from Theorem 3.19. The rightmost inequality in (3.4) was proved in [58], using Theorem 3.1. Note that these results are more accurate than their Maker-Breaker analogs, mentioned in the Introduction.

Corollary 3.21 *The inequality* (3.3) *does not generally hold, not even in the special case where the threshold bias for the strict game exists.*

Indeed, it was proved in Theorem 3.5 that the threshold bias for the strict connectivity game on K_n exists and satisfies $f_{\mathcal{T}_n} = \lfloor n/2 \rfloor - 1$ (or $f_{\mathcal{T}_n} = \lfloor n/2 \rfloor$ in some special cases).

Proof of Theorem 3.19

We provide Avoider with a winning strategy for the minimum positive degree monotone game. Let $b = \frac{n-1}{\log(n-2)}$ and let $K_n = (V, E)$. At any point of the game, let A denote the set of vertices that have a positive degree in Avoider's graph. For a vertex $v \in V$, let $d_{\mathcal{E}}(v)$ denote the degree of v in Enforcer's graph and let $d_{\mathcal{E}}^*$ denote the average degree of vertices of $V \setminus A$ in Enforcer's graph, that is,

$$d_{\mathcal{E}}^* = \frac{\sum_{v \in V \setminus A} d_{\mathcal{E}}(v)}{|V \setminus A|}.$$

Avoider will make sure that after each of his moves there is no unclaimed edge with both endpoints in A.

In his first move Avoider claims an arbitrary unclaimed edge. Let $M = \{v \in V \setminus A : d_{\mathcal{E}}(v) = \min\{d_{\mathcal{E}}(u) : u \in V \setminus A\}\}$ denote the set of vertices that are of minimum degree in Enforcer's graph amongst all vertices of $V \setminus A$. In every other move, Avoider arbitrarily chooses a vertex of M for which there exists a vertex $w \in A$ such that (v, w) is unclaimed. He then claims all unclaimed edges (v, u) where $u \in A$. If every edge with one endpoint in A and the other in M was already claimed by Enforcer, then Avoider claims an arbitrary unclaimed edge (u, w) and all unclaimed edges in $\{(u, x) : x \in A\} \cup \{(w, x) : x \in A\}$. Note that this is a valid strategy even if the latter set is empty.

Avoider stops following this strategy as soon as some vertex $v \in V \setminus A$ first satisfies $d_{\mathcal{E}}(v) \geq n - 1 - b$. Avoider then claims all the remaining unclaimed edges but the ones incident with some arbitrary vertex $z \in V \setminus A$ for which $d_{\mathcal{E}}(z) \geq n - 1 - b$.

In order for this strategy to succeed, there must be at least two unclaimed edges left in E, and at least two vertices left in $V \setminus A$ after the "stopping rule" comes into effect; we will prove that this is indeed the case. This entails Avoider's win, since after his last move, Enforcer must claim all the remaining b edges incident with z, which thus becomes isolated in Avoider's graph.

We will prove by induction on the number of rounds that $d^*_{\mathcal{E}} \geq \sum_{i=2}^{|A|} \frac{b}{n-i}$ after each but the very last move of Enforcer. This is certainly true after the first round, as then $|A| = 2$ and Enforcer has claimed at least b edges. Each of these edges has at least one endpoint in $V \setminus A$, implying $d^*_{\mathcal{E}} \geq \frac{b}{n-2}$.

Next, assume that the statement holds after Enforcer's lth move. We will prove that it remains valid after his next move. Let k denote the cardinality of A at this point. We distinguish between two cases.

Case 1. There exists a vertex $v \in M$ and a vertex $w \in A$ such that (v, w) is unclaimed.

Avoider plays his $(l + 1)$st move according to the aforementioned strategy and so the cardinality of A is increased by one. Before this move we had $d^*_{\mathcal{E}} \geq \sum_{i=2}^{k} \frac{b}{n-i}$ by the induction hypotheses. Avoider's move did not decrease the value of $d^*_{\mathcal{E}}$ as v was of minimum degree in Enforcer's graph, amongst all vertices of $V \setminus A$. In his $(l + 1)$st move, Enforcer claims at least b edges. By Avoider's strategy, each of these edges has at least one endpoint in $V \setminus A$. Since now $|V \setminus A| = n - (k + 1)$, the value of $d^*_{\mathcal{E}}$ was increased by at least $\frac{b}{n-(k+1)}$. Therefore, after both players have played their $(l + 1)$st move we have $d^*_{\mathcal{E}} \geq \sum_{i=2}^{k+1} \frac{b}{n-i}$. Moreover, by definition, there are at least two unclaimed edges left in E.

Case 2. Every edge (v, w) such that $v \in M$ and $w \in A$, was already claimed by Enforcer.

Then, in particular, $d_{\mathcal{E}}(v) \geq k$ for every vertex $v \in V \setminus A$. We can assume that $k \leq n-2-b$ as otherwise the stopping rule would come into effect; it follows that, in particular, there are at least two unclaimed edges left in E. Avoider's $(l + 1)$st move increases the cardinality of A by either one or two. Hence after this move, we have $d^*_{\mathcal{E}} \geq k \geq |A| - 2 > \sum_{i=2}^{|A|} \frac{b}{n-i}$. The last inequality is true for all cardinalities $|A|$, $3 < |A| \leq n - b$ and sufficiently large n since all the terms in the sum are strictly smaller than 1.

It follows that at the end of every round we have $d^*_{\mathcal{E}} \geq \sum_{i=2}^{|A|} \frac{b}{n-i}$.

Since

$$\sum_{i=2}^{n-2} \frac{b}{n-i} = \sum_{j=2}^{n-2} \frac{b}{j} > b(\log(n-2) - 1),$$

we know that at some point during the game when $|A| \leq n-2$, the average Enforcer-degree in $V \setminus A$ is at least $n-1-b$, so the stopping rule comes into effect and Avoider wins.

□

Remark Our proof of Theorem 3.19 is some sort of a dynamic version of the *Box Game* defined in [30]. It is interesting to note that Avoider's strategy is similar to Maker's strategy in the Box Game which means that a player who wants to claim a complete hyperedge and a player who wants to avoid one, will essentially choose the same strategy.

Remark The proof of Theorem 3.19 as well as the theorem itself when compared with Theorem 3.5, is another example showing that a player in an Avoider-Enforcer game can benefit from claiming more board elements per turn.

3.11 Concluding remarks and open problems

1. **A general criterion for Avoider's win.** It was already indicated in the introduction that our criterion for Avoider's win in the $(p, q, \mathcal{H})$ game, is not effective when q is large. Such a criterion might help us improve our bound on $b^-_{\mathcal{H}_n}$. It would also have a potentially significant impact on traditional Maker-Breaker type games. Often Maker can achieve his goal in some game by creating a pseudo-random graph of a certain edge-density (see e.g. [38], [43] and Chapter 7). Such a graph might need to have a property of "at most" type. Maker could try to achieve such conditions by playing as Avoider and trying to "avoid" occupying too many elements of the winning sets.

2. **A general criterion for monotonicity.** We say that an Avoider-Enforcer game $\mathcal{B}$ is *monotone*, if Enforcer's winning strategy for $(p, q, \mathcal{B})$ implies his win in $(p+1, q, \mathcal{B})$ and $(p, q-1, \mathcal{B})$, while Avoider's winning strategy for $(p, q, \mathcal{B})$ implies his win in $(p-1, q, \mathcal{B})$ and $(p, q+1, \mathcal{B})$.

   Problem 3.22 *Find a sufficient (and possibly also necessary) condition for an Avoider-Enforcer game to be monotone.*

3. **(Asymptotic) monotonicity of $\mathcal{M}_n$ and $\mathcal{H}_n$.** For both the Hamilton cycle game and the perfect matching game there is a significant gap between the corresponding thresholds b^-, b^+ shown in this chapter (the only bounds on $b^+_{\mathcal{M}_n}$ and $b^+_{\mathcal{H}_n}$ that we know are the ones derived from the trivial lower bound on the number of edges in a perfect matching and a Hamilton cycle respectively). It would be interesting to close, or at least to reduce, these gaps.

   We believe that even the following holds.

Conjecture 3.23 *Both the perfect matching game and the Hamilton cycle game are monotone. In particular* $b^-_{\mathcal{M}_n} = b^+_{\mathcal{M}_n}$ *and* $b^-_{\mathcal{H}_n} = b^+_{\mathcal{H}_n}$.

The function $f(n)$ is called an *asymptotic threshold bias* of the game $\mathcal{B}_n$ if both $b^-_{\mathcal{B}_n} = \Theta(f(n))$ and $b^+_{\mathcal{B}_n} = \Theta(f(n))$. If an asymptotic threshold bias exists, that is, if $b^-_{\mathcal{B}_n} = \Theta\left(b^+_{\mathcal{B}_n}\right)$, then the game $\mathcal{B}_n$ is called *asymptotically monotone*.

It would be a significant step towards proving Conjecture 3.23 if one could establish that the perfect matching and Hamilton cycle games are asymptotically monotone and determine the order of magnitude of the asymptotic threshold bias. Recall that currently we do not even know whether Avoider can win $(1, o(n), \mathcal{M}_n)$ or $(1, o(n), \mathcal{H}_n)$.

4. **Strict vs. monotone:** In this chapter, it was proved that strict Avoider-Enforcer games can have a different outcome than monotone Avoider-Enforcer games (even when the strict game is monotone). A Natural question arises: ”is one of the rules “better” than the other”? The monotone rules are of course monotone and thus every monotone game has a threshold bias. Moreover, they yield results that are similar to their Maker-Breaker analogues. This “similarity” phenomenon was extensively studied by Beck [17] in the case of unbiased games (that is, $p = q = 1$). On the other hand, many times Avoider-Enforcer games are used to study Maker-Breaker games (see e.g. [45] and Chapter 2) or discrepancy type games. Clearly in these cases the monotone rules are useless.

Part II

Unbiased games and games on sparse graphs

Chapter 4

Fast and slow winning strategies

4.1 Introduction

Let p and q be positive integers and let $\mathcal{H}$ be a hypergraph. In a $(p, q, \mathcal{H})$ Maker-Breaker game, two players, called Maker and Breaker, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Maker selects p vertices per move and Breaker selects q vertices per move. Maker wins if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Breaker wins. (Sometimes, when there is no risk of confusion, we will omit $\mathcal{H}$ in the notation above, calling a $(p, q, \mathcal{H})$-game simply a (p, q)-game.) For a $(1, 1, \mathcal{H})$ Maker-Breaker game, let $\tau_M(\mathcal{H})$ be the smallest integer t such that Maker can win the game within t moves (if the game is a Breaker's win, then set $\tau_M(\mathcal{H}) = \infty$).

Similarly, in a $(p, q, \mathcal{H})$ Avoider-Enforcer game two players, called Avoider and Enforcer, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Avoider selects p vertices per move and Enforcer selects q vertices per move. Avoider loses if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Enforcer loses. For a $(1, 1, \mathcal{H})$ Avoider-Enforcer game, let $\tau_E(\mathcal{H})$ be the smallest integer t such that Enforcer can win the game within t rounds (if the game is an Avoider's win, then set $\tau_E(\mathcal{H}) = \infty$).

In this chapter, our attention is restricted to games which are played on the edges of the complete graph on n vertices, that is, the vertex set of $\mathcal{H}$ will always be $E(K_n)$. For quite a few Maker-Breaker and Avoider-Enforcer games it is rather easy to determine the winner. For example, in the connectivity game played on the edges of the complete graph K_n on n vertices, Maker can easily construct a spanning tree by the end of the game. The Avoider-Enforcer planarity game, played on the edges of K_n for n sufficiently large, is an even more convincing example – Avoider creates a non-planar graph and thus loses the game in the end, irregardless of his strategy, the prosaic reason being that every graph on n vertices with more than $3n - 6$ edges is non-planar. Thus, for games of this type, a more interesting question to ask is not who wins but rather how long it should take the winner to reach a winning position. This is the type of question we address in this chapter.

We start with providing a brief overview of known and relevant results about fast wins in Maker-Breaker and Avoider-Enforcer games. As an immediate consequence of the result of Lehman [60], Maker has a fast winning strategy in the connectivity game. That is, $\tau_M(\mathcal{T}_n) = n-1$ for every $n \geq 4$, where $\mathcal{T}_n$ is the hypergraph whose hyperedges are the (edge sets of the) spanning trees of K_n. This approach can be easily generalized to a fast winning strategy for Maker in the k-edge-connectivity game. Indeed, if K_n contains $2k$ pairwise edge disjoint spanning trees, then by partitioning them into k pairs and applying Lehman's strategy to each pair we get $\frac{1}{2}kn \leq \tau_M(\mathcal{T}_n^k) \leq k(n-1)$, for every $n \geq 4k$, where $\mathcal{T}_n^k$ is the hypergraph whose hyperedges are the spanning k-edge-connected subgraphs of K_n. The lower bound follows immediately since the minimum degree of a k-connected graph is at least k. In this chapter we substantially reduce the gap between these two bounds. As another immediate consequence of Lehman's result, we get that Enforcer cannot win the Avoider-Enforcer cycle game faster than the trivial bound suggests, that is, $\tau_E(\mathcal{C}_n) = n$, where the hyperedges of $\mathcal{C}_n$ are all the cycles of K_n. A result of Bednarska [18] entails $\tau_M(\mathcal{TB}_n^k) = k-1$, where the hyperedges of $\mathcal{TB}_n^k$ are all the copies of complete binary trees on k vertices in K_n, and $k = o(n)$. In [30], Chvátal and Erdős provide Maker with a fast winning strategy for the $(1, 1, \mathcal{H}_n)$ Hamilton cycle game, showing that $\tau_M(\mathcal{H}_n) \leq 2n$, where $\mathcal{H}_n$ is the hypergraph whose hyperedges are the Hamilton cycles of K_n. In this chapter, we almost completely close the gap between this upper bound and the trivial lower bound of $n+1$. Maker can win the $(1, 1, \mathcal{K}_n^q)$ clique game in a constant (depending on q but not on n) number of moves, that is, $\tau_M(\mathcal{K}_n^q) = f(q)$, where the hyperedges of $\mathcal{K}_n^q$ are the q-cliques of K_n. The best upper bound, $f(q) = O((q-3)2^{q-1})$ is due to Pekeč (see [66]); Beck proved that the exponential dependency on q cannot be avoided, namely $f(q) = \Omega(\sqrt{2}^q)$ (see [8]). Note that Maker's strategy for the clique game provides him with a fast win in the non-planarity game and the non-r-colorability game by building a copy of K_5 and K_{r+1}, respectively (for background on these games, see [45] and Chapter 2).

Some general sufficient conditions for winning Maker-Breaker games and Avoider-Enforcer games were proved in [6] and [42] (see also Chapter 3), respectively. Both are based on the "potential" method of Erdős and Selfridge [36]. These criteria, however, seem not to be very useful for winning quickly, as it is assumed that the game is played until every element of the board is claimed by some player. Nonetheless, using some "fake moves" trick (see [17]), they can be used to get certain, usually rather weak, results.

If Maker wins a $(1, q, \mathcal{H})$ Maker-Breaker game for some positive integer q, then $\tau_M(\mathcal{H}) \leq v(\mathcal{H})/(q+1)$, where $v(\mathcal{H})$ is the number of vertices in $\mathcal{H}$. Indeed, when playing the $(1, 1, \mathcal{H})$ game, Maker can use his winning strategy in the $(1, q, \mathcal{H})$ game. In every round, he imagines that additional $q-1$ arbitrary unclaimed vertices were claimed by Breaker. Whenever Breaker claims a vertex which is already his in Maker's imagination, Maker imagines that another (arbitrary still unclaimed) vertex was claimed by Breaker. Clearly, after all vertices have been claimed (including the ones in Maker's imagination), Maker has already won, and the number of rounds played is $v(\mathcal{H})/(q+1)$. Equivalently, this shows that if Breaker can keep from losing the $(1, 1, \mathcal{H})$ game within t rounds, then he can win the $(1, \frac{v(\mathcal{H})}{t} - 1, \mathcal{H})$ game. It was proved by Beck in [4] that Breaker, playing the $(1, 1, \mathcal{H})$ game on an almost disjoint n-uniform hypergraph $\mathcal{H}$, can keep from losing for at least $(2-\varepsilon)^n$ moves, for any $\varepsilon > 0$. Hence, we can immediately deduce that Breaker can win the $(1, \frac{v(\mathcal{H})}{(2-\varepsilon)^n} - 1)$ game, on any almost disjoint n-uniform hypergraph $\mathcal{H}$ and for every $\varepsilon > 0$. Similarly, if Avoider wins the $(1, q, \mathcal{H})$ game for some positive integer q, then $\tau_E(\mathcal{H}) > v(\mathcal{H})/(q+1)$. Indeed, when playing the $(1, 1, \mathcal{H})$ game, Avoider can use his winning strategy from the $(1, q, \mathcal{H})$ game. Equivalently, this shows that if Enforcer can win the game on $\mathcal{H}$ within t rounds, then he can also win the $(1, \frac{v(\mathcal{H})}{t} - 1, \mathcal{H})$ game.

To conclude, in order to say something non-trivial about the games we analyze, we will have to find winning strategies for Maker and Enforcer that are faster than the known strategies mentioned above (in case they exist).

4.1.1 Fast strategies for Maker and slow strategies for Breaker

We now turn back to the analysis of some concrete games. All games we consider here are played on the edges of the complete graph K_n.

Let $\mathcal{M}_n$ be the hypergraph whose hyperedges are all perfect matchings of K_n (or matchings that cover every vertex but one, if n is odd). Let $\mathcal{D}_n$ be the hypergraph whose hyperedges are all spanning subgraphs of K_n of positive minimum degree. We find the *exact* number of moves that Maker needs, in order to win the $(1, 1, \mathcal{M}_n)$ game and the $(1, 1, \mathcal{D}_n)$ game. Obviously, Maker needs to make at least $\lfloor \frac{n}{2} \rfloor$ moves, as this is the size of a member of $\mathcal{M}_n$. We show that if n is odd, then he does not need more moves, whereas if n is even, then he needs just one more move. A similar result, showing the tightness of the obvious lower bound for the minimum degree game $\mathcal{D}_n$, easily follows.

Theorem 4.1 *(i)*

$$\tau_M(\mathcal{M}_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even} \end{cases}$$

(ii)

$$\tau_M(\mathcal{D}_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

As mentioned earlier, Chvátal and Erdős [30] proved that Maker can win the $(1, 1)$ Hamilton cycle game on K_n within $2n$ rounds. Here we show that for sufficiently large n, Maker can win the $(1, 1)$ Hamilton cycle game within $n + 2$ rounds. This bound is now only 1 away from the obvious lower bound.

Theorem 4.2 *For sufficiently large n,*

$$n + 1 \leq \tau_M(\mathcal{H}_n) \leq n + 2.$$

A corollary of the proof of the previous theorem is that Maker can win the "Hamilton path" game within $n - 1$ moves, which is clearly best possible.

Theorem 4.3 *For sufficiently large n,*

$$\tau_M(\mathcal{HP}_n) = n - 1,$$

where $\mathcal{HP}_n$ is the hypergraph whose hyperedges are all Hamilton paths of K_n.

Let $\mathcal{V}_n^k$ be the hypergraph whose hyperedges are all spanning k-vertex-connected subgraphs of K_n. The classical theorem of Lehman [60] asserts that Maker can build a 1-connected spanning graph in $n-1$ moves. From Theorem 4.2 it follows that Maker can build a 2-vertex-connected spanning graph for the price of spending just 3 more (that is, in $n+2$) moves.

In the following, we obtain a generalization of the latter fact for every $k \geq 3$. As every k-connected graph has minimum degree at least k, Maker needs at least $kn/2$ moves just to build a member of $\mathcal{V}_n^k$ (even if Breaker doesn't play at all). The next theorem shows that this trivial lower bound is asymptotically tight, that is, there is a strategy for Maker to build a k-vertex-connected graph in $kn/2 + o_k(n)$ moves.

Theorem 4.4 *For every fixed $k \geq 3$ and sufficiently large n,*

$$kn/2 \leq \tau_M(\mathcal{V}_n^k) \leq kn/2 + (k+4)(\sqrt{n} + 2n^{2/3}\log n).$$

An easy consequence of Theorems 4.1, 4.2 and 4.4, is that for every fixed $k \geq 1$ Maker can build a graph with minimum degree at least k within $(1+o(1))kn/2$ moves. This is clearly asymptotically optimal.

4.1.2 Slow strategies for Avoider and fast strategies for Enforcer

In the Avoider-Enforcer non-planarity game, Avoider loses the game as soon as his graph becomes non-planar. Clearly, Enforcer can win this game within $3n-5$ moves no matter how he plays; that is, $\tau_E(\mathcal{NP}_n) \leq 3n-5$, where $\mathcal{NP}_n$ is the hypergraph whose hyperedges are all non-planar subgraphs of K_n. On the other hand, Avoider can keep from losing for $\frac{3}{2}n-3$ moves by simply fixing any triangulation and claiming its edges arbitrarily for as long as possible.

The following theorem asserts that the trivial upper bound is essentially tight, that is, Avoider can refrain from building a non-planar graph for at least $(3-o(1))n$ moves. More precisely,

Theorem 4.5

$$\tau_E(\mathcal{NP}_n) > 3n - 28\sqrt{n}.$$

In the Avoider-Enforcer non-k-coloring game $\mathcal{NC}_n^k$, Avoider loses the game as soon as his graph becomes non-k-colorable. Avoider can play for at least $(1-o(1))\frac{(k-1)n^2}{4k}$ moves without losing by simply fixing a copy of the k-partite Turán-graph and claiming half of its edges. On the other hand, it is not hard to see that the game is an Enforcer's win if it is played until the end (see Theorem 3.12 from Chapter 3), so Avoider will lose after at most $\frac{1}{2}\binom{n}{2} \approx \frac{n^2}{4}$ moves. In our next theorem we essentially close the gap between the two bounds for the case $k=2$ (the "non-bipartite game"). We also improve the trivial lower bound and establish the order of magnitude of the second order term of $\tau_E(\mathcal{NC}_n^2)$.

Theorem 4.6

$$\frac{n^2}{8} + \frac{n-2}{12} \leq \tau_E(\mathcal{NC}_n^2) \leq \frac{n^2}{8} + \frac{n}{2} + 1.$$

Next, we look at two Avoider-Enforcer games that turn out to be of similar behavior. In the game $\mathcal{D}_n$ Enforcer wins as soon as the minimum degree in Avoider's graph becomes positive, and in the game $\mathcal{T}_n$ Enforcer wins as soon as Avoider's graph becomes connected and spanning. Enforcer wins both games (see [42] and Chapter 3), entailing $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n}{2}$. On the other hand, Avoider can choose an arbitrary vertex v, and, for as long as possible, claim only edges which are not incident with v, implying $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) > \frac{1}{2}\binom{n-1}{2}$. This determines the first order term for both parameters. In the following theorem we determine the second order term and the order of magnitude of the third.

Theorem 4.7

$$\frac{1}{2}\binom{n-1}{2} + \left(\frac{1}{4} - o(1)\right)\log n < \tau_E(\mathcal{D}_n) \leq \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1.$$

The rest of this chapter is organized as follows: in Section 4.2 we prove Theorems 4.1, 4.2 and 4.4. In Section 4.3 we prove Theorems 4.5, 4.6 and 4.7. Finally, in Section 4.4 we present some open problems.

4.1.3 Preliminaries

For the sake of simplicity and clarity of presentation, we omit floor and ceiling signs whenever these are not crucial. Some of our results are asymptotic in nature and, whenever necessary, we assume that n is sufficiently large. Throughout this chapter, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [31]. In particular, we use the following: for a graph G, denote its set of vertices by $V(G)$, and its set of edges by $E(G)$. Moreover, let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a graph $G = (V, E)$ and a set $A \subseteq V$ denote by $G[A]$ the subgraph of G induced by A. Let $N_G(A) = \{u \in V : \exists w \in A, (u, w) \in E\}$ be the neighborhood of A in G and let $\Gamma_G(A) = N_G(A) \setminus A$ be the external-neighborhood of A in G. Sometimes, when there is no risk of confusion, we abbreviate $N_G(A)$ to $N(A)$ and $\Gamma_G(A)$ to $\Gamma(A)$.

4.2 Maker-Breaker games

In our definition of Maker-Breaker games, Maker starts the game. In the following, whenever proving a result of the form $\tau_M(\mathcal{H}) \leq a$, we will assume that Breaker starts the game (thus proving a statement which is stronger than the one asserted in the corresponding theorem).

4.2.1 Building a perfect matching fast

Proof of Theorem 4.1.

Assume first that n is even. Obviously Maker needs at least $n/2$ edges to build a perfect matching. In fact he will need at least one more, as Breaker, seeing the first $n/2-1$ moves of Maker, can occupy the unique edge (if no such edge exists, then our claim immediately follows) which would extend Maker's graph into a perfect matching. Hence $\tau_M(\mathcal{M}_n) \geq \frac{n}{2}+1$.

In the following we assume that Breaker starts the game and give a strategy for Maker to build his perfect matching in $\frac{n}{2}+1$ moves. A *round* of the game consists of a move by Breaker and a counter move by Maker. A vertex is considered *bad*, if it is isolated in Maker's graph but not in Breaker's graph.

We will provide Maker with a strategy to ensure that for every $3 \leq r \leq \frac{n}{2}$, the following three properties hold after his rth move:

(a) Maker's edges form a forest consisting of $r-1$ components: a path uvw of length two and $r-2$ paths of length one;

(b) every isolated vertex of Maker's graph is adjacent to neither u nor w in Breaker's graph;

(c) there are at most two bad vertices.

First, let us see that, if these properties hold after Maker's $\frac{n}{2}$th move, then Maker wins the perfect matching game on his next move. Observe that by property (a) after the $\frac{n}{2}$th move of Maker there is exactly one isolated vertex z in Maker's graph, which, by property (b), is connected to neither u nor w in Breaker's graph. Hence, no matter which edge Breaker claims in his $(\frac{n}{2}+1)$st move, Maker will be able to respond by claiming either (u,z) or (w,z). After that move Maker's graph is a spanning forest consisting of a path of length three and $\frac{n}{2}-2$ paths of length one; obviously such a graph contains a perfect matching.

Next, we prove that for every $n \geq 6$, Maker can maintain properties $(a)-(c)$. First, it is easy to see that Maker can execute his first three moves such that these three properties hold.

We will prove that on his rth move, where $\frac{n}{2} \geq r > 3$, Maker can select two vertices that are isolated in his graph and connect them by an edge, while ensuring that, right after his move, properties (b) and (c) hold. Note that this strategy automatically ensures that property (a) holds as well.

Let I_r be the set of vertices which are isolated in Maker's graph after the rth round. Property (a) ensures that $|I_r| = n-(2r-1)$ and property (c) implies that there are at most two vertices in I_r which are not isolated in Breaker's graph; in particular there is at most one edge in Breaker's graph spanned by I_r. Assume that the rth round, where $r \leq n/2-1$, has just ended, then $|I_r| \geq 3$.

In case Breaker claims an edge of the form (x,u) or (x,w) where $x \in I_r$, then Maker responds by claiming an edge (x,y) where $y \in I_r$. Such a vertex y for which the edge (x,y) was not previously claimed by Breaker always exists as only one of Breaker's edges is spanned by I_r, and there are at least three vertices in I_r. Since the vertex x will not be bad at the end of the $(r+1)$st round, the number of bad vertices does not increase and property (c) remains valid. Property (b) will also remain valid because the only new vertex which could dissatisfy it, x, is not isolated in Maker's graph anymore.

If Breaker does not claim an edge of the form (x, u) or (x, w), where $x \in I_r$, then Maker responds by claiming an edge with both endpoints in I_r such that property (c) remains valid. This can easily be done as there are at most two edges of Breaker with both endpoints in I_r, and $|I_r| \geq 3$. Property (b) was not affected by Breaker's move.

This concludes our description of Maker's strategy and the proof if n is even.

If n is odd, then Maker's strategy is essentially the same as his strategy for even n (in fact it is a little simpler). The main difference is that property (b) is redundant, property (a) is replaced with:

(a') After Maker's rth round, his graph is a matching with r edges,

and we don't need to consider separately, Maker's first three moves. We omit the straightforward details.

As for the positive minimum degree game, it is clear that $\tau_M(\mathcal{D}_n) \geq \lfloor n/2 \rfloor + 1$. Furthermore, if n is even, then by part (i) of Theorem 4.1 we get $\tau_M(\mathcal{D}_n) \leq \tau_M(\mathcal{M}_n) = n/2 + 1$. If n is odd, then Maker can build a matching that covers all vertices but one in $\lfloor n/2 \rfloor$ rounds, and then claim an arbitrary edge incident with the last remaining isolated vertex. Hence, we get $\tau_M(\mathcal{D}_n) = \lfloor n/2 \rfloor + 1$ as claimed.

□

4.2.2 Building a Hamilton cycle fast

Proof of Theorem 4.2.

In the proof, we use the method of Pósa rotations (see [67]). Let $P_0 = (v_1, v_2, \ldots, v_l)$ be a path of maximum length in a graph G. If $1 \leq i \leq l-2$ and (v_l, v_i) is an edge of G then $P' = (v_1, v_2, \ldots, v_i, v_l, v_{l-1}, \ldots, v_{i+1})$ is also of maximum length. It is called a *rotation* of P_0 with *fixed endpoint* v_1 and *pivot* v_i. The edge (v_i, v_{i+1}) is called the *broken* edge of the rotation. We can then, in general, rotate P' to get more maximum length paths.

We will assume that Breaker starts the game. A *round* consists of a move by Breaker and a counter move by Maker. Assume first that n is even. Maker's strategy is divided into three stages.

In the first stage, Maker builds a perfect matching with one additional edge, that is, he builds a path of length 3 and $(n-4)/2$ paths of length 1. From the proof of Theorem 4.1 we know that Maker can do this in $n/2 + 1$ moves.

In the second stage, which lasts exactly $n/2 - 2$ rounds, Maker connects endpoints of the paths in his graph. In each move he connects two paths to form one longer path. Hence, in each round he decreases the number of paths by one, and thus, by the end of the second stage he will have a Hamilton path.

For every $0 \leq i \leq n/2 - 3$, let B_i' be the subgraph of Breaker's graph, induced by the endpoints of Maker's paths, just after the $(i+1)$st move of Breaker in the second stage (recall that Breaker starts the second stage). Let B_i be the graph obtained from B_i' by removing all edges (x,y) such that x and y are endpoints of the same path of Maker. The unclaimed edges $(x,y) \in \binom{V(B_i)}{2}$, for which x and y are endpoints of different paths of Maker are called *available*.

The first move of Maker in this stage is somewhat artificial, thinking ahead about stage three. Let $w \in V(B_0)$ be a vertex of highest degree in Breaker's graph. On his first move of the second stage Maker claims an arbitrary available edge incident with w. Such an edge exists if n is large enough, since Breaker has $n/2 + 2$ edges, while there are $n - 2$ endpoints in $V(B_0)$. Note that for any two vertices $z', z'' \in V(B_1)$, the sum of the degrees of z' and z'' in Breaker's graph is at most $n/3 + 4$ (we will use this observation only in stage three).

Maker's goal is now the following: he will make sure that $e(B_i) \leq v(B_i) - 1$ for every $1 \leq i \leq n/2 - 3$. This easily holds for $i = 1$ provided n is large enough. Assume that the statement holds for some $1 \leq i \leq n/2 - 4$ and let us prove that Maker can claim an available edge while ensuring that $e(B_{i+1}) \leq v(B_{i+1}) - 1$.

Case 1.j. (for every $0 \leq j \leq 3$). $e(B_i) \leq v(B_i) - 1 - j$ and there is an available edge incident with at least $3 - j$ edges of B_i. Maker claims this edge entailing $e(B_{i+1}) \leq e(B_i) - (3 - j) + 1 \leq v(B_i) - 3 = v(B_{i+1}) - 1$.

Case 2. There is a vertex v of degree at least 3 in B_i. Hence by Case 1.0 we can assume that there is no available edge incident with v, that is, the degree of v in B_i is exactly $v(B_i) - 2$ (recall that there are no edges in B_i between the endpoints of the same path of Maker). Note that by the induction hypothesis there is at most one edge in B_i which is not incident with v. Since $i \leq n/2 - 4$, $v(B_i) \geq 6$, and so v has at least four neighbors in B_i.

Assume first that every edge of B_i is incident with v, entailing $e(B_i) = v(B_i) - 2$. Among the four neighbors of v there has to be at least one available edge. This edge is incident with two edges of Breaker and so Case 1.1 applies.

Suppose now that there is an edge of B_i which is not incident with v. One of its endpoints z is a neighbor of v. Hence, since $v(B_i) \geq 6$, there must exist an available edge between z and another neighbor of v; thus Case 1.0 applies.

Case 3. The maximum degree of B_i is at most 2. Hence every connected component of B_i is either a path or a cycle. By Case 1.3 we can assume that $e(B_i) > v(B_i) - 4$. If $e(B_i) = v(B_i) - 3$, then by Case 1.2 Maker can claim any available edge which is incident with some edge of Breaker. If $e(B_i) = v(B_i) - 2$, then there is a vertex x of degree 2, since $v(B_i) \geq 6$. By Case 1.1 Maker can claim any available edge which is incident with x. Finally, if $e(B_i) = v(B_i) - 1$, then again there is a vertex x of degree 2. Moreover, there is an available edge incident with x whose other endpoint y is a non-isolated vertex in B_i (such a non-isolated vertex exists, since $v(B_i) \geq 6$ and $e(B_i) = v(B_i) - 1$). Maker claims the edge (x,y) and Case 1.0 applies.

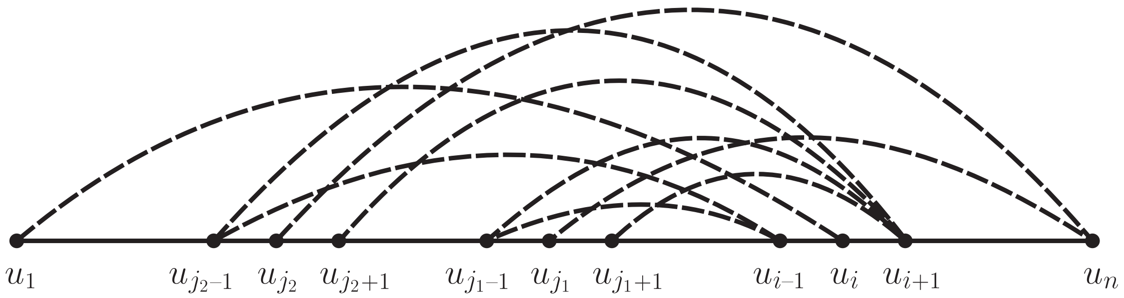

Figure 4.1: Dashed edges are unclaimed by Breaker.

This means that after $n/2-3$ moves in the second stage Maker has successfully built a spanning forest consisting of two paths such that Breaker's graph $B_{n/2-3}$ on the four endpoints of these two paths satisfies $e(B_{n/2-3}) \leq v(B_{n/2-3})-1$. Hence, there exists at least one available edge in $B_{n/2-3}$. Maker claims this edge, thus creating his Hamilton path.

In the third stage, Maker uses Pósa rotations to close his Hamilton path $u_1, u_2, \ldots, u_n$ to a Hamilton cycle. Let u_i, u_{j_1}, u_{j_2} be three vertices on this path such that $i-1 > j_1+1 > j_2+1$ and, just before Maker's first move in this stage, none of the edges (u_1, u_i), (u_{j_1}, u_n), (u_{j_2}, u_n), (u_{i+1}, u_{j_1-1}), (u_{i-1}, u_{j_1-1}), (u_{i+1}, u_{j_1+1}), (u_{i+1}, u_{j_2-1}), (u_{i-1}, u_{j_2-1}), (u_{i+1}, u_{j_2+1}) were previously claimed by Breaker (see Figure 4.1). In his first move of the third stage, Maker claims the edge (u_1, u_i). In his next move, Breaker cannot claim both (u_{j_1}, u_n) and (u_{j_2}, u_n). Assume without loss of generality that he does not claim (u_{j_1}, u_n). In his next move Maker claims (u_{j_1}, u_n), and then he claims either (u_{i+1}, u_{j_1-1}) or (u_{i-1}, u_{j_1-1}) or (u_{i+1}, u_{j_1+1}) (Breaker cannot neutralize these three simultaneous threats with only two edges). This yields a Hamilton cycle. Note that stage three lasts exactly 3 rounds.

It remains to prove that the three vertices u_i, u_{j_1}, u_{j_2} with the desired properties exist. Recall that, by Maker's first move in the second stage, we have $deg_{B_1}(u_1) + deg_{B_1}(u_n) \leq n/3+4$. In the second and third stages Breaker adds $n/2$ more edges, entailing $deg_{B_{n/2-3}}(u_1) + deg_{B_{n/2-3}}(u_n) \leq 5n/6+4$. Hence, for sufficiently large n, there are at least $n/7$ vertices u_k such that neither (u_1, u_k) nor (u_k, u_n) was claimed by Breaker. Thus there are at least $n^2/200$ pairs of vertices u_i, u_j such that $i-1 > j+1$ and both (u_1, u_i) and (u_j, u_n) were not claimed by Breaker. Moreover, Breaker has only $O(n)$ edges and every edge (u_p, u_q) he claims affects at most four of the pairs (u_i, u_j), namely $(u_{p-1}, u_{q-1}), (u_{p-1}, u_{q+1}), (u_{p+1}, u_{q-1})$ and (u_{p+1}, u_{q+1}). Hence, there exist two such pairs u_i, u_{j_1} and u_i, u_{j_2}.

If n is odd, then the proof is essentially the same, with just a few small technical changes:

1. The first stage lasts $\lfloor n/2 \rfloor + 1$ rounds and, when it ends, Maker has one path of length 2 and $(n-3)/2$ paths of length 1.

2. The second stage lasts exactly $\lceil n/2 \rceil - 2$ rounds.

3. In B_0 there are $n-1$ vertices and at most $\lfloor n/2 \rfloor + 2$ edges.

□

4.2.3 Building a k-connected graph fast

Proof of Theorem 4.4.

Let $K_n = (V, E)$ where $V = \{1, 2, \ldots, n\}$. Assume first that n is even and let $m = kn/2$. We will present a random strategy for Maker, which enables him to build a k-vertex-connected graph within $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ rounds, with positive probability. This, however, will imply the existence of a deterministic strategy for Maker with the same outcome.

Before we start with a detailed description of Maker's strategy, we give an short overview of his actions. The game consists of two stages (it is possible that the second stage will not take place). In the first stage most of Maker's moves are used for building a graph which is "not far" from being a random k-regular graph. The motivation for this approach is that random k-regular graphs are known to be k-vertex-connected a.s. (for more on random regular graphs, the reader is referred to [25], [50] and [77]). In this stage Maker also has to watch out for Breaker's maximum degree growing too large; he will handle this by momentarily abandoning the creation of the pseudo-random graph in order to occupy some edges incident with the "dangerous vertex" (that is, a vertex of high degree in Breaker's graph). In the second stage, Maker occupies some more edges to neutralize possible damage to his pseudo-random graph, caused by Breaker during the first stage.

Before the beginning of the game, Maker does the following. With every $1 \leq i \leq n$, he associates a set $W_i = \{i_1, i_2, \ldots, i_k\}$ of "copies" of i, the sets being pairwise disjoint. Maker then draws uniformly at random a perfect matching P of the $2m$ elements of $W = \bigcup_{i=1}^{n} W_i$. Let $S = ((a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m))$ be an arbitrary ordering of the matched pairs. Note that the selection of the perfect matching P, can be done equivalently by choosing the pairs one at a time. That is, Maker repeatedly draws a pair randomly, uniformly on all unmatched elements of W. Sometimes this point of view is more convenient for our analysis. If $a_r \in W_i$ and $b_r \in W_j$, then we say that the pair (a_r, b_r) corresponds to the edge (i, j). Clearly, different pairs can correspond to the same edge, and so it is possible to get parallel edges. Furthermore, it is possible that $\{a_r, b_r\} \subseteq W_i$ and so the pair (a_r, b_r) corresponds to the loop (i, i). Thus the pairing P corresponds to a k-regular multi-graph. We will discard loops and parallel edges and thus obtain a simple graph of maximum degree at most k.

A vertex $i \in V$ will be called *dangerous* if its degree in Breaker's graph is at least $k\sqrt{n}$. As soon as such a vertex appears, Maker "treats" it immediately (this process will be described in the following paragraph). Throughout the game, let D denote the set of all dangerous vertices which were already "treated". Before the game starts we set $D = \emptyset$.

Stage 1: During this stage, if there are no dangerous vertices outside D, then Maker claims edges of K_n according to the ordering S (note that the matching P and its ordering S are not known to Breaker). That is, let r be the smallest positive integer such that the pair (a_r, b_r) was not considered by Maker before. Maker then claims the edge (i, j), where $(a_r, b_r) = (i_p, j_q)$ for some $1 \leq i, j \leq n$ and $1 \leq p, q \leq k$. If $i = j$ or the edge (i, j) was previously claimed, either by him or by Breaker, then Maker skips his turn (that is, he claims an arbitrary edge which will not be considered in the analysis) and the pair (a_r, b_r) is marked a *failure*. As soon as some $u \in V$ becomes dangerous (if there are several dangerous vertices, then Maker picks one arbitrarily), Maker suspends the above mentioned strategy and plays as follows. He arbitrarily picks $2k+8$ vertices $w_1, w_2, \ldots, w_{2k+8} \notin D$ such that the edges (u, w_j) are unclaimed for every $1 \leq j \leq 2k+8$ and, at that point, no w_j is adjacent in Maker's graph to any vertex in D. This is always possible since the first stage lasts less than $kn/2$ moves, so there can be at most $\sqrt{n}$ dangerous vertices. Handling each such vertex takes $k+4$ moves, so any dangerous vertex, when handled, has degree at most $k\sqrt{n} + (k+4)\sqrt{n}$ in Breaker's graph, and every vertex which is not in D has degree at most $k+1$ in Maker's graph. During his next $k+4$ moves, Maker claims some $k+4$ edges from the set $\{(u, w_1), (u, w_2), \ldots, (u, w_{2k+8})\}$. He then labels u treated, adds it to D and returns to his usual strategy. The first stage ends as soon as every dangerous vertex is treated and all but $kn^{2/3}$ pairs of S are considered by Maker. The last $kn^{2/3}$ pairs of S are also considered to be failures.

Lemma 4.8 *During the first stage there are at most $n^{2/3} \log n$ failures almost surely.*

Proof of Lemma 4.8: It is well-known that for every fixed k, an n-vertex k-regular multi-graph that corresponds to a random pairing, almost surely contains at most $n^{2/3}$ loops and parallel edges (see e.g. [50]). Hence, it suffices to bound from above the number of failures that correspond to edges that were previously claimed by Breaker. Throughout the first stage, there are at most $\sqrt{n}$ vertices in D. Hence, after considering at most $kn/2 - kn^{2/3}$ pairs of S, there are at least $n^{2/3} < 2n^{2/3} - \sqrt{n} - (k+4)\sqrt{n}$ vertices of degree strictly smaller than k in Maker's graph. It follows that at any point during the first stage there are at least $\binom{n^{2/3}}{2} - kn/2$ edges available for Maker to continue his configuration (following S). Since Breaker has claimed at most $kn/2$ edges to this point, the probability that any specific pair (a_i, b_i) corresponds to an edge that was previously claimed by Breaker (here we view S as if it was built sequentially) is at most

$$\frac{kn/2}{\binom{n^{2/3}}{2} - kn/2} \leq \frac{2k}{n^{1/3}}.$$

Let F be the random variable that counts the number of the first $kn/2 - kn^{2/3}$ pairs of S, that correspond to edges that were previously claimed by Breaker. Then

$$\mathbb{E}(F) \leq \frac{kn}{2} \cdot \frac{2k}{n^{1/3}} \leq k^2 n^{2/3}.$$

Using Markov's inequality we obtain

$$Pr(F \geq n^{2/3}(\log n - k - 1)) = o(1).$$

It follows that almost surely throughout Stage 1 there are at most $n^{2/3} \log n$ failures ($n^{2/3}(\log n - k - 1)$ for hitting Breaker's edges, $n^{2/3}$ for loops and parallel edges and $kn^{2/3}$ for the last $kn^{2/3}$ pairs of S), which proves the statement of the lemma. $\square$

Let $G_1 = (V, E)$ denote the graph that Maker has built in the first stage, following his random strategy. Let X be the set of all vertices of $V \setminus D$ that are incident with at least one edge, that corresponds to a failure pair, and let $V = V_1 \cup V_2$ be a partition of V, where $V_1 = D \cup X$. Observe that each vertex of V_2 is incident with k random edges of the random graph defined by P. We can thus derive expansion properties of subsets of V_2 from those of the random k-regular graph. This is done in the following claim.

Claim 4.9 *The following holds almost surely. There exists a constant $c > 0$ such that if $A \subseteq V_2$ and $|A| < c \log n$, then $|\Gamma(A)| \geq (k-2)|A|$, and if $A \subseteq V_2$, $B \subseteq V \setminus A$, where $c \log n \leq |A| \leq |B|$ and $|B| \geq n - k - |A|$, then there is an edge between a vertex of A and a vertex of B. Moreover, if $|A| = 1$, then $|\Gamma(A)| \geq k$, and if $|A| = 2$, then $|\Gamma(A)| \geq 2k - 3$.*

The proof of Claim 4.9 is essentially the same as the proof of Theorem 7.32 from [25]. We omit the straightforward details.

As we already mentioned, since we are looking at a finite, perfect information game with no chance moves, it follows that Maker has a deterministic strategy to build $G_1 = (D \cup X \cup V_2, E)$ within $kn/2 + (k+4)\sqrt{n}$ moves, such that $|D| \leq \sqrt{n}$, $|X| \leq 2n^{2/3} \log n$, and V_2 satisfies the properties described in Claim 4.9.

Stage 2: For every $u \in X$, Maker arbitrarily picks $2k+8$ vertices $w_1^u, w_2^u, \ldots, w_{2k+8}^u \in V \setminus N(D)$, such that the edges (u, w_j^u) are unclaimed for every $1 \leq j \leq 2k+8$ and $\{w_1^u, w_2^u, \ldots, w_{2k+8}^u\} \cap \{w_1^v, w_2^v, \ldots, w_{2k+8}^v\} = \emptyset$ for every $u \neq v \in X$. This is possible as $|X| \leq 2n^{2/3} \log n$, $|D| \leq \sqrt{n}$, and each vertex in X has $n - o(n)$ unclaimed edges incident with it, as $X \cap D = \emptyset$. Using an obvious pairing strategy, Maker claims $k+4$ of the edges (u, w_j^u) for every $u \in X$.

Let G_M denote the graph built by Maker during the entire game. We claim that it is k-vertex-connected. Assume for the sake of contradiction, that a small set separates G_M, that is, $V = A \cup S \cup B$, where $1 \leq a = |A| \leq |B|$, $|S| = s < k$ and there are no edges between A and B in G_M. If $a \leq 5$ and $x \in A \cap V_1$, then by Maker's strategy $|(\Gamma(A) \cup A) \setminus \{x\}| \geq |\Gamma(x)| \geq k+4 > |(A \cup S) \setminus \{x\}|$ which is a contradiction as $(\Gamma(A) \cup A) \setminus \{x\} \subseteq (A \cup S) \setminus \{x\}$. On the other hand, if $A \cap V_1 = \emptyset$, then $|\Gamma(A)| \geq k$ by Claim 4.9 (recall that $k \geq 3$). Hence, from now on we assume that $6 \leq a < c \log n$. If $|A \cap V_1| \geq a/4$, then by Maker's strategy $|N(A \cap V_1)| \geq (k+4)a/4 > a + k \geq |A \cup S|$ which is a contradiction as $N(A \cap V_1) \subseteq A \cup S$. Otherwise, $|A \cap V_1| < a/4$ and so by Claim 4.9 we have $|\Gamma(A \cap V_2)| \geq (k-2)3a/4 \geq a/4 + k > |(A \cap V_1) \cup S|$, where the second inequality follows since $a \geq 6$ and $k \geq 3$. Again, this is a contradiction.

If n is odd, then Maker plays as follows. He arbitrarily picks some vertex u and then plays two disjoint games in parallel. One is on the board $\{(u, v) : v \in V \setminus \{u\}\}$, which is played until he claims exactly k of its elements, and the other is on $K_n[V \setminus \{u\}] \cong K_{n-1}$, where Maker plays according to the above strategy. It is easy to see that the resulting graph is k-vertex-connected (adding a vertex to a k-connected graph and then connecting it to k arbitrary vertices of the graph produces a k-connected graph).

Finally, note that by Maker's strategy and by Lemma 4.8, in both stages Maker plays at most $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ moves. $\square$

4.3 Avoider-Enforcer games

4.3.1 Keeping the graph planar for long

Proof of Theorem 4.5.

We begin by introducing some terminology. Let v be a vertex, and let S be a set of vertices. Let $N_{\mathcal{A}}(v, S)$ denote the set of neighbors of v in Avoider's graph, belonging to S. Similarly, let $N_{\mathcal{E}}(v, S)$ denote the set of neighbors of v in Enforcer's graph, belonging to S.

We will provide Avoider with a strategy for keeping his graph planar for at least $3n - 28\sqrt{n}$ rounds. The strategy consists of three stages.

Before the game starts, we partition the vertex set

$$V(K_n) = \{v_1\} \dot{\cup} \{v_2\} \dot{\cup} A \dot{\cup} N_{1,1} \dot{\cup} N_{1,2} \dot{\cup} N_{2,1} \dot{\cup} N_{2,2},$$

such that $|N_{1,1}| = |N_{1,2}| = |N_{2,1}| = |N_{2,2}| = \sqrt{n} - 1$ and $|A| = n - 4\sqrt{n} + 2$.

In the first stage, Avoider claims edges according to a simple pairing strategy. For every vertex $a \in A$, we pair up the edges (a, v_1) and (a, v_2). Whenever Enforcer claims one of the paired edges, Avoider immediately claims the other edge of that pair. If Enforcer claims an edge which does not belong to any pair, then Avoider claims the edge (a, v_1), for some $a \in A$, for which neither (a, v_1) nor (a, v_2) were previously claimed. He then removes the pair $(a, v_1), (a, v_2)$ from the set of considered edge pairs.

The first stage ends as soon as Avoider connects every $a \in A$ to either v_1 or v_2. Note that, at that point, Avoider's graph consists of two vertex-disjoint stars centered at v_1 and v_2, and the isolated vertices in $N_{1,1} \cup N_{1,2} \cup N_{2,1} \cup N_{2,2}$. Hence, during the first stage, Avoider has claimed exactly $n - 4\sqrt{n} + 2$ edges. Define $A_1 := N_{\mathcal{A}}(v_1, A)$, and $A_2 := N_{\mathcal{A}}(v_2, A)$.

Before the second stage starts, we pick four vertices $n_{1,1} \in N_{1,1}$, $n_{1,2} \in N_{1,2}$, $n_{2,1} \in N_{2,1}$ and $n_{2,2} \in N_{2,2}$, such that $|N_{\mathcal{E}}(n_{i,j}, A)| \leq \sqrt{n}$ for every $i, j \in \{1, 2\}$. Clearly, such a choice of vertices is possible as the total number of edges Enforcer has claimed during the first stage is $n - 4\sqrt{n} + 2 < \sqrt{n} \cdot (\sqrt{n} - 1)$. Define $G_1 := N_{\mathcal{E}}(n_{1,1}, A_1) \cup N_{\mathcal{E}}(n_{1,2}, A_1)$, and $G_2 := N_{\mathcal{E}}(n_{2,1}, A_2) \cup N_{\mathcal{E}}(n_{2,2}, A_2)$. Note that $|G_1| \leq 2\sqrt{n}$, $|G_2| \leq 2\sqrt{n}$, and $|N_{\mathcal{E}}(n_{1,1}, A_1 \setminus G_1)| = |N_{\mathcal{E}}(n_{1,2}, A_1 \setminus G_1)| = |N_{\mathcal{E}}(n_{2,1}, A_2 \setminus G_2)| = |N_{\mathcal{E}}(n_{2,2}, A_2 \setminus G_2)| = 0$.

Using a pairing strategy similar to the one used in the first stage, Avoider connects each vertex of $A_1 \setminus G_1$ to either $n_{1,1}$ or $n_{1,2}$, and each vertex of $A_2 \setminus G_2$ to either $n_{2,1}$ or $n_{2,2}$. More precisely, for every $a \in A_1 \setminus G_1$ we pair up the edges $(a, n_{1,1})$ and $(a, n_{1,2})$, and for every $a \in A_2 \setminus G_2$ we pair up edges $(a, n_{2,1})$ and $(a, n_{2,2})$. Avoider then proceeds as in the first stage.

The second stage ends as soon as Avoider connects every $a \in A_1 \setminus G_1$ to either $n_{1,1}$ or $n_{1,2}$, and every $a \in A_2 \setminus G_2$ to either $n_{2,1}$ or $n_{2,2}$. We define $A_{1,1} := N_{\mathcal{A}}(n_{1,1}, A_1)$, $A_{1,2} := N_{\mathcal{A}}(n_{1,2}, A_1)$, $A_{2,1} := N_{\mathcal{A}}(n_{2,1}, A_2)$ and $A_{2,2} := N_{\mathcal{A}}(n_{2,2}, A_2)$. Since $|A_{1,1}| + |A_{1,2}| = |A_1| - |G_1|$, $|A_{2,1}| + |A_{2,2}| = |A_2| - |G_2|$ and $|A_1| + |A_2| = |A|$, we infer that the number of edges Avoider has claimed in the second stage is at least $n - 8\sqrt{n}$. Note that during the first two stages Avoider did not claim any edge with both endpoints in one of the sets $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, $A_{2,2}$.

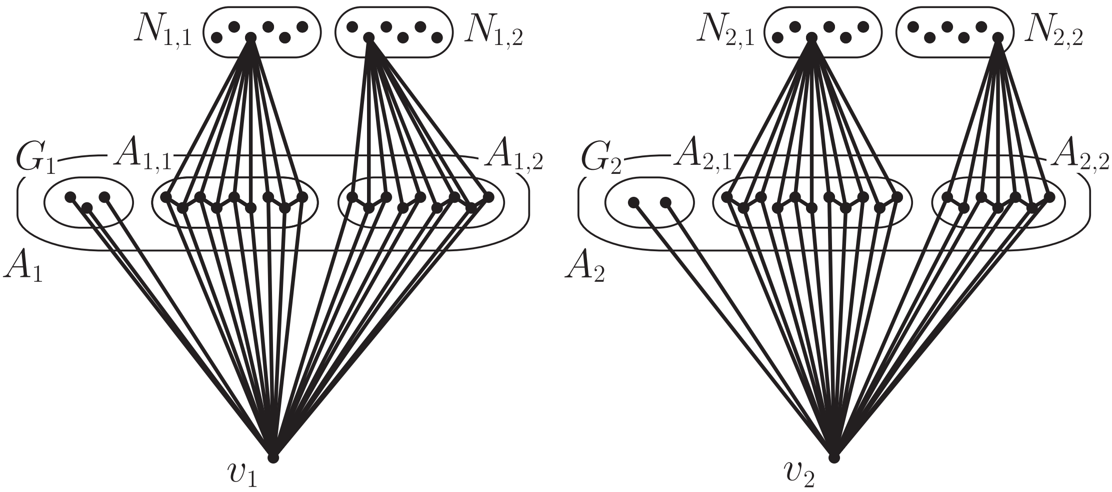

Figure 4.2: Avoider's graph.

In the third stage, Avoider claims only edges with both endpoints contained in the sets $A_{i,j}$, for some $i, j \in \{1, 2\}$. His goal in this stage is to build a "large" linear forest in $A_{1,1}$. (A *linear forest* is a vertex-disjoint union of paths.) In the beginning of the third stage, Avoider's graph induced on the vertices of $A_{1,1}$ is empty, that is, it consists of $|A_{1,1}|$ paths of length 0 each. For as long as possible, Avoider claims edges that connect endpoints of two of his paths in $A_{1,1}$, creating a longer path. When this is no longer possible, every edge that connects endpoints of two different paths must have been previously claimed by Enforcer. Since the total number of edges that Enforcer has claimed so far is at most $3n$, the number of paths of Avoider in $A_{1,1}$ is at most $2\sqrt{n}$. Hence, Avoider has claimed at least $|A_{1,1}| - 2\sqrt{n}$ edges to this point of the third stage.

Similarly, Avoider builds a "large" linear forest in $A_{1,2}$, $A_{2,1}$, and finally $A_{2,2}$, all in the same way. Thus, the total number of edges he claims during the third stage is at least

$$\begin{aligned}\sum_{i,j\in\{1,2\}} (|A_{i,j}| - 2\sqrt{n}) &\geq |A_1| - |G_1| + |A_2| - |G_2| - 8\sqrt{n} \\ &\geq |A| - 12\sqrt{n} \\ &\geq n - 16\sqrt{n}.\end{aligned}$$

The total number of edges claimed by Avoider during the entire game is therefore at least $(n - 4\sqrt{n}) + (n - 8\sqrt{n}) + (n - 16\sqrt{n}) = 3n - 28\sqrt{n}$. Moreover, at the end of the third stage (which is also the end of the game), Avoider's graph is the pairwise edge disjoint union of two stars, four other graphs - each being a subgraph of a union of K_{2,n_i} and a linear forest which is restricted to one side of the bipartition (see Figure 4.2). Clearly, such a graph is planar.

□

4.3.2 Avoiding an odd cycle for long

Proof of Theorem 4.6.

Forcing an odd cycle fast. First, we provide Enforcer with a strategy that will force Avoider to claim the edges of an odd cycle during the first $\frac{n^2}{8} + \frac{n}{2} + 1$ moves. In every stage of the game, each connected component of Avoider's graph is a bipartite graph with a unique bipartition of the vertices (we stop the game as soon as Avoider is forced to close an odd cycle). In every move, Enforcer's primary goal is to claim an edge which connects two opposite sides of the bipartition of one of the connected components of Avoider's graph. If no such edge is available, then Enforcer claims an arbitrary edge, and that edge is marked as "possibly bad". Clearly, in his following move Avoider cannot play inside any of the connected components of his graph either, and so he is forced to merge two of his connected components (that is, he has to claim an edge (x, y) such that x and y are in different connected components of his graph). As the game starts with n connected components, this situation can occur at most $n - 1$ times.

Therefore, when Avoider is not able to claim any edge without creating an odd cycle, his graph is bipartite, and all of Enforcer's edges, except some of the "possibly bad" ones, are compatible with the bipartition of Avoider's graph. The total number of edges that were claimed by both players to this point is at most $\frac{n^2}{4} + n - 1$, and so the total number of moves Avoider has played in the entire game is at most $\frac{n^2}{8} + \frac{n}{2} + 1$.

Avoiding an odd cycle for long. Next, we provide Avoider with a strategy for keeping his graph bipartite for at least $\frac{n^2}{8} + \frac{n-2}{12}$ rounds. For technical reasons we assume that n is even; however, a similar statement holds for odd n as well. During the game Avoider will maintain a family of ordered pairs (V_1, V_2), where $V_1, V_2 \subseteq V(K_n)$, $V_1 \cap V_2 = \emptyset$ and $|V_1| = |V_2|$, which he calls *bi-bunch*es. We say that two bi-bunches (V_1, V_2) and (V_3, V_4) are disjoint if $(V_1 \cup V_2) \cap (V_3 \cup V_4) = \emptyset$. At any point of the game, Avoider calls a vertex *untouched* if it does not belong to any bi-bunch and all the edges incident with it are unclaimed. During the entire game, we will maintain a partition of the vertex set $V(K_n)$ into a number of pairwise disjoint bi-bunches, and a set of untouched vertices.

Avoider starts the game with n untouched vertices and no bi-bunches. In every move, his primary goal is to claim an edge *across* some existing bi-bunch, that is, an edge (x, y) where $x \in V_1$ and $y \in V_2$ for some bi-bunch (V_1, V_2). If no such edge is available, then he claims an edge joining two untouched vertices x and y, introducing a new bi-bunch $(\{x\}, \{y\})$. If he is unable to do that either, then he claims an edge connecting two bi-bunches, that is, an edge (x, y) such that there exist two bi-bunches (V_1, V_2) and (V_3, V_4) with $x \in V_1$ and $y \in V_3$. He then replaces these two bi-bunches with a single new one $(V_1 \cup V_4, V_2 \cup V_3)$.

Whenever Enforcer claims an edge (x, y) such that neither x nor y belong to any bi-bunch, we introduce a new bi-bunch $(\{x, y\}, \{u, v\})$, where u and v are arbitrary untouched vertices. If at that point of the game there are no untouched vertices (clearly this can happen at most once), then the new bi-bunch is just $(\{x\}, \{y\})$. If Enforcer claims an edge (x, y) such that there is a bi-bunch (V_1, V_2) with $x \in V_1$ and y is untouched, then the bi-bunch (V_1, V_2) is replaced with $(V_1 \cup \{y\}, V_2 \cup \{u\})$, where u is an arbitrary untouched vertex. Finally, if Enforcer claims an edge (x, y) such that there are bi-bunches (V_1, V_2) and (V_3, V_4) with $x \in V_1$ and $y \in V_3$, than these two bi-bunches are replaced with a single one $(V_1 \cup V_3, V_2 \cup V_4)$. Note that by following his strategy, and updating the bi-bunch partition as described, Avoider's graph will not contain an edge with both endpoints in the same side of a bi-bunch at any point of the game.

Observe that the afore-mentioned bi-bunch maintenance rules imply the following. If Enforcer claims an edge (x, y), such that before that move x was an untouched vertex, then the edge (x, y) will be contained in the same side of some bi-bunch, that is, after that move there will be a bi-bunch (V_1, V_2) with $x, y \in V_1$ (unless x and y were the last two isolated vertices).

Assume that in some move Avoider claims an edge (x, y), such that before that move x was an untouched vertex. It follows from Avoider's strategy that y was untouched as well, and there were no unclaimed edges across a bi-bunch at that point. Thus, in his next move, Enforcer will also be unable to claim an edge across a bi-bunch and so, by the bi-bunch maintenance rules for Enforcer's moves, the edge he will claim in that move will have both its endpoints in the same side of some bi-bunch.

By the previous paragraphs, we conclude that after every round in which at least one of the players claims an edge which is incident with an untouched vertex (which is not the next to last untouched vertex), the edge Enforcer claims in this round will be contained in the same side of some bi-bunch. By the bi-bunch maintenance rules, during every round the number of untouched vertices is decreased by at most 6. Hence, by the time all but two vertices are not untouched at least $(n-2)/6$ edges of Enforcer will be contained in the same side of a bi-bunch. Therefore, when Avoider can no longer claim an edge without creating an odd cycle, both players have claimed together all the edges of a balanced bipartite graph which is in compliance with the bi-bunch bipartition, and at least another $(n-2)/6$ edges. This gives a total of at least $\frac{n}{2} \cdot \frac{n}{2} + (n-2)/6$ edges claimed, which means that at least $\frac{n^2}{8} + \frac{n-2}{12}$ rounds were played to that point. □

4.3.3 Keeping an isolated vertex for long

Proof of Theorem 4.7.

Clearly $\tau_E(\mathcal{D}_n) \leq \tau_E(\mathcal{T}_n)$ and so it suffices to prove that $\tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ and that, $\tau_E(\mathcal{D}_n) > \frac{1}{2}\binom{n-1}{2} + (1/4 - \varepsilon)\log n$ for every $\varepsilon > 0$ and sufficiently large n.

Forcing a spanning tree fast. Starting with the former inequality, we provide Enforcer with a strategy to force Avoider to build a connected spanning graph within $\frac{1}{2}\binom{n-1}{2}+2\log_2 n+1$ rounds. At any point of the game, we call an edge that was not claimed by Avoider *safe*, if both its endpoints belong to the same connected component of Avoider's graph. An edge which is not safe and was not claimed by Avoider is called *dangerous*. Denote by G_D the graph consisting of dangerous edges claimed by Enforcer. We will provide Enforcer with a strategy to make sure that, throughout the game, the maximum degree of the graph G_D does not exceed $4k$, where $k = \log_2 n$.

Assuming the existence of such a strategy, the assertion of the theorem readily follows. Indeed, assume for the sake of contradiction that after $\frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ rounds have been played (where Enforcer follows the afore-mentioned strategy), Avoider's graph is disconnected. Let $C_1, \ldots, C_r$, where $r \geq 2$ and $|C_1| \leq \ldots \leq |C_r|$, be the connected components in Avoider's graph at that point. By Enforcer's strategy, the maximum degree of the graph G_D does not exceed $4k$. Hence, the number of edges claimed by both players to this point does not exceed

$$\sum_{i=1}^{r} \binom{|C_i|}{2} + 4k \sum_{i=1}^{r-1} |C_i|.$$

Assuming that $r \geq 2$ and n is sufficiently large, this sum above attains its maximum for $r = 2$, $|C_1| = 1$ and $|C_2| = n - 1$; that is, the sum is bounded from above by $\binom{n-1}{2} + 4\log_2 n$ - a contradiction.

Now we provide Enforcer with a strategy for making sure that, throughout the game, the maximum degree of the graph G_D does not exceed $4k$. In every move, if there exists an unclaimed safe edge, Enforcer claims it (if there are several such edges, Enforcer claims one arbitrarily). Hence, whenever Enforcer claims a dangerous edge, Avoider has to merge two connected components of his graph in the following move, and the number of Avoider's connected components is decreased by one. We will use this fact to estimate the number of dangerous edges at different points of the game.

When all edges within each of the connected components of Avoider's graph are claimed, Enforcer has to claim a dangerous edge. His strategy for choosing dangerous edges is divided into two phases. The first phase is divided into k stages. In the ith stage Enforcer will make sure that the maximum degree of the graph G_D is at most $2i$; other than that, he claims dangerous edges arbitrarily. He proceeds to the following stage only when it is not possible to play in compliance with this condition. Let c_i, $i = 1, \ldots, k$, denote the number of connected components in Avoider's graph after the ith stage. Let $c_0 = n$, be the number of components at the beginning of the first stage. During the ith stage, a vertex v is called *saturated*, if $d_{G_D}(v) = 2i$. Note that at the beginning of the first stage the maximum degree of G_D is $2 \cdot 0 = 0$.

We will prove by induction that $c_i \leq n2^{-i} + 2i$, for all $i = 0, 1, \ldots, k$. The statement trivially holds for $i = 0$.

Next, assume that $c_j \leq n2^{-j} + 2j$, for some $0 \leq j < k$. At the beginning of the $(j+1)$st stage Avoider's graph has exactly c_j connected components, and at the end of this stage it has exactly c_{j+1} components. It follows that during this stage Avoider merged two components of his graph $c_j - c_{j+1}$ times. Hence, Enforcer has not claimed more than $c_j - c_{j+1}$ dangerous edges during the $(j+1)$st stage. As the maximum degree of the graph G_D before this stage was $2j$, the number of saturated vertices at the end of the $(j+1)$st stage is at most $c_j - c_{j+1}$. It follows that there are at least $n - (c_j - c_{j+1})$ non-saturated vertices at this point.

The non-saturated vertices must be covered by at most $2(j+1)$ connected components of Avoider's graph. Indeed, assume for the sake of contradiction that there are non-saturated vertices $u_1, u_2, \ldots, u_{2j+3}$ and connected components $U_1, U_2, \ldots, U_{2j+3}$, such that $u_p \in U_p$ for every $1 \leq p \leq 2j+3$. Since $deg_{G_D}(u_p) \leq 2j+1$ for every $1 \leq p \leq 2j+3$, it follows that there must exist an unclaimed edge (u_r, u_s) for some $1 \leq r < s \leq 2j+3$, contradicting the fact that the $(j+1)$st stage is over. Therefore, there are at least $c_{j+1} - 2(j+1)$ connected components in Avoider's graph that do not contain any non-saturated vertex. Clearly every such component has size at least one, entailing $(c_{j+1} - 2j - 2) + (n - c_j + c_{j+1}) \leq n$. Applying the inductive hypothesis we get $c_{j+1} \leq c_j/2 + j + 1 \leq n2^{-(j+1)} + 2(j+1)$. This completes the induction step.

It follows, that at the end of the first phase, after the kth stage, the number of connected components in Avoider's graph, is at most $c_k \leq n2^{-k} + 2k \leq 2k+1$.

In the second phase, whenever Enforcer is forced to claim a dangerous edge, he claims one arbitrarily. Since at the beginning of the second phase, there are at most $2k+1$ connected components in Avoider's graph, Enforcer will claim at most $2k$ dangerous edges during this phase.

It follows that at the end of the game, the maximum degree in G_D will be at most $4k$, as claimed.

Keeping an isolated vertex for long. Fix $\varepsilon > 0$ and set $l := \frac{1-4\varepsilon}{2} \log n$. We provide Avoider with a strategy to keep an isolated vertex in his graph for at least $\frac{1}{2}\binom{n-1}{2} + \frac{l}{2}$ rounds.

Throughout the game, Avoider's graph will consist of one connected component, which we denote by C, and $n - |C|$ isolated vertices. A vertex $v \in V(K_n) \setminus C$ is called *bad*, if there is an even number of unclaimed edges between v and C; otherwise, v is called *good*.

For every vertex $v \in V(K_n)$ let $d_{\mathcal{E}}(v)$ denote the degree of v in Enforcer's graph. If at any point of the game there exists a vertex $v \in V(K_n) \setminus C$ such that $d_{\mathcal{E}}(v) \geq l$, then Avoider simply proceeds by arbitrarily claiming edges which are not incident with v, for as long as possible. The total number of rounds that will be played in that case is at least $\frac{1}{2}\binom{n-1}{2} + \frac{l}{2}$, which proves the theorem. We will show that Avoider can make sure that such a vertex $v \in V(K_n) \setminus C$, with $d_{\mathcal{E}}(v) \geq l$, will appear before the order of his component C reaches $n - l\varepsilon^{-1} - 1$. Hence, from now on, we assume that $|C| \leq n - l\varepsilon^{-1} - 2$.

Whenever possible, Avoider will claim an edge with both endpoints in C. If this is not possible, he will join a new vertex to the component, that is, he will connect it by an edge to an arbitrary vertex of C. Note that this is always possible. Indeed, assume that every edge between C and $V(K_n) \setminus C$ was already claimed by Enforcer. If $|C| \geq l$ then there exists a vertex $v \in V(K_n)$ such that $d_{\mathcal{E}}(v) \geq l$ and so we are done by the previous paragraph. Otherwise, $|C| < l$ and thus, until this point, Enforcer has claimed at most $l^2 < l(n-l)$ edges. As for the way he chooses this new vertex, we consider three cases. Let $\overline{\overline{d}}$ denote the average degree in Enforcer's graph, taken over all the vertices of $V(K_n) \setminus C$, that is,

$$\overline{\overline{d}} := \frac{\sum_{v \in V(K_n) \setminus C} d_{\mathcal{E}}(v)}{n - |C|}.$$

Throughout the case analysis, C and $\overline{\overline{d}}$ represent the values as they are just before Avoider makes his selection.

1. There exists a vertex $v \in V(K_n) \setminus C$, such that $d_{\mathcal{E}}(v) \leq \overline{\overline{d}} - 1$.

   Avoider joins v to his component C. Then $|C|$ increases by one, and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n-|C|)\overline{\overline{d}} - (\overline{\overline{d}} - 1)}{n - |C| - 1} = \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

2. Every vertex $v \in V(K_n) \setminus C$ satisfies $d_{\mathcal{E}}(v) > \overline{\overline{d}} - 1$, and $\overline{\overline{d}} < \lfloor \overline{\overline{d}} \rfloor + 1 - \varepsilon$.

   Let D denote the set of vertices $u \in V(K_n) \setminus C$ such that $d_{\mathcal{E}}(u) = \lfloor \overline{\overline{d}} \rfloor$. Note that there must be at least $\varepsilon(n - |C|)$ vertices in D. We distinguish between the following two subcases.

   (a) There is a good vertex in D. Avoider joins it to his component C (if there are several good vertices, then he picks one arbitrarily). Since v was a good vertex, Enforcer must claim at least one edge (x, y) such that $x \notin C \cup \{v\}$, before Avoider is forced again to join another vertex to his component. After this move of Enforcer $|C|$ is (still) increased by (just) one, and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n-|C|)\overline{\overline{d}} - \lfloor \overline{\overline{d}} \rfloor + 1}{n - |C| - 1} \geq \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

   (b) All vertices in D are bad. Knowing that $d_{\mathcal{E}}(v) \leq l-1$ for all vertices $v \in V(K_n) \setminus C$, and $|C| \leq n - l\varepsilon^{-1} - 2$, we have

$$\max_{v \in D} d_{\mathcal{E}}(v) = \lfloor \overline{\overline{d}} \rfloor < l - 1 + 2\varepsilon \leq \varepsilon(n - |C|) - 1 \leq |D| - 1$$

   and hence there have to be two vertices $u, w \in D$ such that (u, w) is unclaimed. Avoider joins u to his component C, and thus w becomes good. If Enforcer, in his next move, claims an edge (w, v) for some $v \in C$, then $|C|$ is increased by one and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n-|C|)\overline{\overline{d}} - \lfloor \overline{\overline{d}} \rfloor + 1}{n - |C| - 1} \geq \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

   Otherwise, in his next move Avoider joins w to C. Since w was good, then, as in the previous subcase, Enforcer will be forced to claim an edge (x, y) such that $x \notin C \cup \{w\}$. After that move of Enforcer, we will have that $|C|$ is still increased just by two and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n-|C|)\overline{\overline{d}} - \lfloor \overline{\overline{d}} \rfloor - \lfloor \overline{\overline{d}} \rfloor + 1}{n - |C| - 2} \geq \overline{\overline{d}} + \frac{1}{n - |C| - 2}.$$

3. Every vertex $v \in V(K_n) \setminus C$ satisfies $d_{\mathcal{E}}(v) > \overline{\overline{d}} - 1$, and $\overline{\overline{d}} \geq \lfloor \overline{\overline{d}} \rfloor + 1 - \varepsilon$.

   Let D denote the set of vertices in $V(K_n) \setminus C$ with degree either $\lfloor \overline{\overline{d}} \rfloor$ or $\lfloor \overline{\overline{d}} \rfloor + 1$. Clearly, $|D| \geq \frac{1}{2}(n - |C|)$. We distinguish between the following two subcases.

   (a) There is a good vertex in D. Similarly to subcase 2(a), Avoider joins that vertex to his component C, and after Enforcer claims some edge with at least one endpoint outside C, we have that $|C|$ is increased by one and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n - |C|)\overline{\overline{d}} - (\overline{\overline{d}} + \varepsilon) + 1}{n - |C| - 1} = \overline{\overline{d}} + \frac{1 - \varepsilon}{n - |C| - 1}.$$

   (b) All vertices in D are bad. Similarly to subcase 2(b), Avoider can find two vertices in D such that the edge between them is unclaimed. He joins them to his component C, one after the other. After Enforcer claims some edge with at least one endpoint outside C, we have that $|C|$ increased by two and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n - |C|)\overline{\overline{d}} - (\overline{\overline{d}} + \varepsilon) - (\overline{\overline{d}} + \varepsilon) + 1}{n - |C| - 2} = \overline{\overline{d}} + \frac{1 - 2\varepsilon}{n - |C| - 2}.$$

It follows that in all cases the value of $\overline{\overline{d}}$ grows by at least $\frac{1-2\varepsilon}{n-|C|-1}$, whenever $|C|$ grows by at most 2. Hence, when the size of C reaches $n - l\varepsilon^{-1} - 2$, we have

$$\begin{aligned}
\overline{\overline{d}} &\geq \sum_{i=2}^{n/2-\frac{1}{2\varepsilon}l-1} \frac{1-2\varepsilon}{n-2i-1} \\
&\geq \frac{1-2\varepsilon}{2} \sum_{i=4}^{n-l\varepsilon^{-1}-2} \frac{1}{n-i-1} \\
&\geq \frac{1-2\varepsilon}{2} \left(\sum_{i=1}^{n-5} \frac{1}{i} - \sum_{i=1}^{l\varepsilon^{-1}} \frac{1}{i} \right) \\
&\geq \frac{1-3\varepsilon}{2} \left(\log n - \log(l\varepsilon^{-1}) \right) \\
&\geq l,
\end{aligned}$$

which concludes the proof of the theorem.

□

4.4 Concluding remarks and open problems

- It was proved in Theorem 4.4 that Maker can win the $(1,1)$ k-vertex-connectivity game on K_n within $kn/2 + o(n)$ moves. It would be interesting to decide whether the

$o(n)$ term can be replaced with some function of k, if not for this game, then for the k-edge-connectivity game or the minimum-degree-k game.

- It was proved in Theorem 4.6 that $\tau_E(\mathcal{NC}_n^2) \leq \frac{n^2}{8} + \Theta(n)$. For $k \geq 3$, we know just the trivial bounds $\frac{(k-1)n^2}{4k} \leq \tau_E(\mathcal{NC}_n^k) \leq \frac{1}{2}\binom{n}{2}$. It would be interesting to close, or at least reduce, the gap between these bounds. It seems reasonable that, as in the case $k = 2$, the truth is closer to the trivial lower bound, and maybe $\tau_E(\mathcal{NC}_n^k) \leq (1 + o(1))\frac{(k-1)n^2}{4k}$ for every $k \geq 3$.

- It was proved in Theorem 4.7 that $\tau_E(\mathcal{T}_n)$ and $\tau_E(\mathcal{D}_n)$ are "almost the same". This is reminiscent of the well-known property of random graphs, that the hitting time of being connected and the hitting time of having minimum positive degree are a.s. the same, and it motivates us to raise the following conjecture.

  Conjecture 4.10 $\tau_E(\mathcal{D}_n) = \tau_E(\mathcal{T}_n)$.

- It would be interesting to obtain good estimates on $\tau_E(\mathcal{M}_n)$ and $\tau_E(\mathcal{H}_n)$.

Chapter 5

Games on random graphs

5.1 Introduction

Let p and q be two positive integers, X a finite set, and $\mathcal{F} \subseteq 2^X$ a hypergraph. In the positional game $(X, \mathcal{F}, p, q)$ two players take turns claiming previously unclaimed elements of X. In every round, the first player claims p elements, and then the second player responds by claiming q elements (in the very end of the game it may happen that the number of elements of X left unclaimed is strictly less than the number of vertices the player about to play is expected to claim; in this case that player claims all the remaining elements). The set X is called the "board", and p and q are the biases of the first and second player, respectively. For the purposes of this paper $\mathcal{F}$ is assumed to be monotone increasing. In a *Maker/Breaker-type* positional game, the two players are called Maker and Breaker and $\mathcal{F}$ is referred to as the family of winning sets. Maker wins the game if the subset of X he claims by the end of the game (that is, when every element of the board has been claimed by one of the players) is a winning set, that is, an element of $\mathcal{F}$; otherwise Breaker wins. Since $\mathcal{F}$ is monotone increasing, Maker wins if and only if he occupies all the vertices of an inclusion-minimal element of $\mathcal{F}$. Sometimes, if there is no risk of confusion, we use the notation $\mathcal{F}$ for the family of inclusion-minimal members of $\mathcal{F}$ as well.

The study of positional games on the edge set of a (complete) graph was initiated by Lehman [60] who, in particular, proved that Maker can easily win the $(E(K_n), \mathcal{T}_n, 1, 1)$ game, where the family $\mathcal{T}_n$ consists of the edge-sets of all connected and spanning subgraphs of K_n (by "easily" we mean that he can do so within $n-1$ moves, which is clearly optimal). Chvátal and Erdős [30] suggested to "even out the odds" by giving Breaker more power, that is, by increasing his bias. They determined that the connectivity game $(E(K_n), \mathcal{T}_n, 1, b)$ is won by Maker even when the bias b of Breaker is as large as $cn/\log n$, for some small constant $c > 0$, whereas for another constant $C > 0$, Breaker wins the game if his bias is at least $Cn/\log n$. They also showed that the $(E(K_n), \mathcal{H}_n, 1, 1)$ Hamiltonicity game, in which Maker's goal is to build a Hamilton cycle (that is, the family $\mathcal{H}_n$ of winning sets consists of the edge-sets of all Hamiltonian subgraphs of K_n), is won by Maker for sufficiently large n. Moreover, they conjectured that in fact Maker can win the $(E(K_n), \mathcal{H}_n, 1, b)$ game for some b that tends to infinity with n. This was proved by Bollobás and Papaioannou [27], who showed that Maker wins Hamiltonicity against a bias of $O(\log n/\log\log n)$. Later, Beck [9] gave a winning strategy for Maker against a bias of $O(n/\log n)$. Very recently Krivelevich and Szabó [58] derived a tighter estimate on the maximal bias b for which Maker still wins the Hamiltonicity game.

Following [70] we give Breaker more power, not by increasing his bias, but by "thinning out" the board before the game starts. Formally, let $(X,\mathcal{H})$ be a hypergraph and let $0 \leq p \leq 1$ be a real number. We define $(X_p,\mathcal{H}_p)$ to be the hypergraph whose set of vertices X_p is obtained from X by removing every vertex of X with probability $1-p$, independently for each vertex, and whose set of hyperedges is $\mathcal{H}_p = \{A \in \mathcal{H} \ : A \subseteq X_p\}$. Note that $(X_p,\mathcal{H}_p)$ is actually a probability space of hypergraphs. Looking at the $(X_p,\mathcal{H}_p,1,1)$ game, we can discuss the probability that Maker (Breaker) wins the game.

The *threshold probability* $p_{\mathcal{F}_n}$ for the family of games $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is defined to be the probability for which an almost sure Breaker's win turns into an almost sure Maker's win, that is,

$$Pr[(X_p,(\mathcal{F}_n)_p,1,1) \text{ is a Breaker's win}] \to 1 \text{ for } p = o(p_{\mathcal{F}_n}),$$

and

$$Pr[(X_p,(\mathcal{F}_n)_p,1,1) \text{ is a Maker's win}] \to 1 \text{ for } p = \omega(p_{\mathcal{F}_n}),$$

when $n \to \infty$. Such a threshold $p_{\mathcal{F}_n}$ exists [28], as being a Maker's win is a monotone increasing property.

In this paper, the random hypergraph $(X_p,(\mathcal{F}_n)_p)$ is generated by choosing the edges of the complete graph K_n on n vertices independently and with probability $p = p(n)$. Clearly, the obtained board is nothing else but the binomial random graph $G(n,p)$. A typical question we address is then the following: given a monotone graph property Q (Hamiltonicity, existence of a perfect matching, etc.), what is the threshold probability $p = p(n,Q)$ at which the Maker-Breaker game, played on the edges of $G(n,p)$, in which Maker's goal is to build a subgraph admitting property Q, turns from an almost sure Breaker's win to an almost sure Maker's win (we say that a random graph $G(n,p)$ possesses some property P almost surely, or a.s. for brevity, if $\lim_{n\to\infty} Pr[G(n,p) \text{ has } P] = 1$)? It should be stressed that the probabilistic part of our questions and statements refers to the creation of a random board; once a graph G is generated according to the $G(n,p)$ model, the game is completely deterministic and is either a Maker's win or a Breaker's win.

In [70] the threshold probability for the connectivity game and the perfect matching game was determined. Moreover, it was proved that the threshold probability for the Hamiltonicity game satisfies $\frac{\log n}{n} \leq p_{\mathcal{H}_n} \leq \frac{\log n}{\sqrt{n}}$, with the conjecture that $p_{\mathcal{H}_n} = \frac{\log n}{n}$. This was verified in [69]. Here we strengthen this result and show that the property that Maker wins the Hamiltonicity game has a *sharp* threshold.

Theorem 5.1 *There exists a constant $c' > 0$ such that Maker a.s. wins the $(1,1)$ Hamiltonicity game on $G(n, \frac{\log n + (\log\log n)^{c'}}{n})$.*

This statement is obviously very close to being best possible, as $G(n, \frac{\log n + 3\log\log n - \omega(1)}{n})$, where the $\omega(1)$ term tends to infinity with n arbitrarily slowly, has a.s. at least three vertices of degree at most three each, and so Breaker easily wins by forcing Maker to build a graph with minimum degree at most one.

It is instructive to compare Theorem 5.1 with known results on Hamiltonicity of random graphs. Komlós and Szemerédi [53] proved that if $p = p(n) = \frac{\log n + \log\log n + \omega(1)}{n}$, where $\omega(1)$ is any function tending to infinity arbitrarily slowly with n, then $G(n,p)$ is almost surely Hamiltonian; this estimate is easily seen to be essentially tight. Thus, Theorem 5.1 can be considered as a strengthening (with a slightly weaker bound) of the above stated classical result: for $p(n) = (1+\varepsilon)\frac{\log n}{n}$ not only $G(n,p)$ contains a.s. a Hamilton cycle, but Maker is able to build one while playing against an adversary.

For a graph H, let $\mathcal{F}_H$ be the set of all copies of H in K_n. In the *k-clique-game* $\mathcal{F}_{K_k}$ Maker's goal is to build a complete subgraph on k vertices. In [70] the exponent of the main factor of the threshold-probability $p_{\mathcal{F}_{K_k}}$ for the k-clique-game with constant k was essentially determined. For $k \geq 4$ it was proved that for every $\varepsilon > 0$,

$$n^{-\frac{2}{k+1}-\varepsilon} \leq p_{\mathcal{F}_{K_k}} \leq n^{-\frac{2}{k+1}}.$$

Here the exponent $\frac{2}{k+1}$ is the reciprocal of the so-called 2-density of K_k. The *(maximum) 2-density* of an arbitrary graph H on at least three vertices, defined by

$$m_2(H) = \max_{\substack{H' \subseteq H \\ v(H') \geq 3}} \frac{e(H') - 1}{v(H') - 2},$$

is a well-known parameter in random graph theory. For example, $n^{-1/m_2(H)}$ is the threshold probability for the property of being a graph, every edge of which is contained in a copy of H. Intuitively this means that above this probability, the copies of H are "densely and uniformly distributed" in $G(n,p)$. Then it is not very surprising that Maker is able to win the H-game, played on the edges of $G(n,p)$.

Theorem 5.2 *Let H be a graph containing a cycle. There exists a real number $c_0 > 0$ such that for $p > \frac{1}{c_0} n^{-\frac{1}{m_2(H)}}$ almost surely Maker wins the H-game $\mathcal{F}_H$ on $G(n,p)$.*

It turns out that the methods from [70] for dealing with the clique game can be generalized to the H-game (for certain graphs H). As in the case of cliques K_k for $k \geq 4$, for some graphs H, we are able to prove a lower bound for the threshold probability of the H-game, which essentially matches the upper bound stated in Theorem 5.2. Let $dgn(G) = \max\{\delta(G') : G' \subseteq G\}$ denote the *degeneracy* of G.

Theorem 5.3 *Let H be a graph satisfying $m_2(H) \leq dgn(H) - \frac{1}{2}$. Then, for an arbitrarily small $\varepsilon > 0$ and for $p \leq n^{-\frac{1}{m_2(H)-\varepsilon}}$, Breaker a.s. has a winning strategy for the H-game on $G(n,p)$*

This theorem covers the case of cliques and even more. It is a well known (and easily verifiable) fact that $dgn(H) \geq m_2(H)$, and thus actually $dgn(H) \geq \lceil m_2(H) \rceil$, for every graph H on at least three vertices. Also, $m_2(H) \leq d - \frac{1}{2}$ for every d-regular graph H with $d \geq 3$. We thus derive:

Corollary 5.4 *The conclusion of the above theorem is valid for every d-regular graph H, $d \geq 3$. It is also valid for all graphs H for which $m_2(H) - \lfloor m_2(H) \rfloor \in (0, \frac{1}{2}]$.*

Note that Theorem 5.3 is not applicable to $H = K_3$ as $m_2(K_3) = dgn(K_3) = 2$. In fact, in [70] it was shown that the K_3-game is somewhat of an anomaly in the sense that Theorem 5.2 **can** be strengthened for it. It was proved that the threshold probability for the triangle game is $n^{-\frac{5}{9}}$. Note that $\frac{5}{9} > \frac{1}{2} = m_2(K_3)^{-1}$.

In [70] it was asked for which graphs H we have $p_{\mathcal{F}_H} = \tilde{\Theta}\left(n^{-1/m_2(H)}\right)$ and for which graphs Theorem 5.2 can be improved.

For an arbitrary tree $T \neq K_2$ we have $m_2(T) = 1$ and $dgn(T) = 1$, which means that Theorem 5.3 cannot be applied. We show that Theorem 5.2 can be improved for trees.

Theorem 5.5 *Let $T \neq K_2$ be an arbitrary tree. There exists an $\varepsilon(T) > 0$ such that a.s. Maker wins the game $\mathcal{F}_T$ on $G(n,p)$ for $p \geq n^{-1-\varepsilon(T)}$.*

Note that if $T \neq K_2$ is a matching, then $m_2(T) = \frac{1}{2}$; so the threshold $p_{\mathcal{F}_T}$ is again $n^{-1/m_2(T)}$, as it "should be".

We obtain relatively accurate estimates for the threshold probability in special tree-games, like the path-game and the star-game. We show that $\varepsilon(P_d)$ is exponential in $-d$, while $\varepsilon(S_d)$ is linear in $1/d$. It would be interesting to determine $\varepsilon(T)$ more precisely for other trees.

The rest of this chapter is organized as follows: in Section 5.2 we mention some known results and notation that will be used in the following sections. In Section 5.3 we prove Theorem 5.1, in Section 5.4 we prove Theorems 5.2 and 5.3 as well as several other related results. Finally, in Section 5.5 we present some open problems.

5.2 Preliminaries

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. All of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [31].

For a graph G, let $V(G)$ and $E(G)$ denote its set of vertices and edges respectively; and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a set $A \subseteq V(G)$ let $E_G(A)$ denote the set of edges of G with both endpoints in A, and let $e_G(A) = |E_G(A)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A,B)$ denote the set of edges of G with one endpoint in A and the other in B, and let $e_G(A,B) = |E_G(A,B)|$. For $S \subseteq V(G)$ let $N_G(S) = \{u \in V(G) \setminus S : \exists v \in S, (u,v) \in E(G)\}$ denote the set of neighbors of S in $V(G) \setminus S$. For a vertex $w \in V(G) \setminus S$ let $d_G(w,S) = |\{u \in S : (u,w) \in E(G)\}|$ denote the number of vertices of S that are adjacent to w in G. We abbreviate $d_G(w, V \setminus \{w\})$ to $d_G(w)$ which denotes the degree of w in G. The minimum degree of a vertex in G is denoted by $\delta(G)$. Often, when there is no risk of confusion, we discard G in the notation above.

For a graph G define

$$d(G) = \frac{e(G)}{v(G)},$$

$$m(G) = \max_{H \subseteq G} d(H),$$

$$d_2(G) = \frac{e(G) - 1}{v(G) - 2},$$

and

$$m_2(G) = \max_{\substack{H \subseteq G \\ v(H) \geq 3}} d_2(H),$$

where in the definition of $d_2(G)$ we assume that $v(G) \geq 3$. A graph G satisfying $m(G) = d(G)$ is called *balanced*, and a graph G satisfying $m_2(G) = d_2(G)$ is called *2-balanced*.

Let T be a rooted tree with root r. The *down-degree* of a vertex $v \in V(T) \setminus \{r\}$ is $\underline{d}(v) = d(v) - 1$. The down-degree of the root r is $\underline{d}(r) = d(r)$. The *depth* $\nu(T)$ of the tree T is equal to the maximum length of a path in T of which r is an endpoint, where the length of a path is equal to the number of its edges. A vertex $v \in V(T)$ is said to be on the ith level, if there exists a path of length i from v to r in T; the root r is said to be on the 0th level.

The following classical theorem, due to Erdős and Selfridge [36], is a useful sufficient condition for Breaker's win in an unbiased game $\mathcal{F}$.

Theorem 5.6 *Let $(X, \mathcal{F})$ be an arbitrary hypergraph. If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Breaker has a winning strategy in the $(X, \mathcal{F}, 1, 1)$ game.

As a generalization, we can look at games in which the goal of Maker is to claim several (instead of one) winning sets. Formally, Maker wins the game if he claims all elements of at least c winning sets from $\mathcal{F}$. Equivalently, this is the game $(X, \{\cup_{B \in F} B : F \in \binom{\mathcal{F}}{c}\}, 1, 1)$, whose winning sets are all possible unions of c sets from $\mathcal{F}$.

Theorem 5.7 *([6, 20]) If for a positive integer c we have*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{c}{2},$$

then Breaker has a winning strategy in the $(X, \{\cup_{B \in F} B : F \in \binom{\mathcal{F}}{c}\}, 1, 1)$ game.

5.3 Hamiltonian cycles one-on-one

In this section we are going to prove Theorem 5.1.

Let

$$p = p(n) = \frac{\log n + (\log\log n)^{c'}}{n} \ ,$$

and let

$$k = k(n) = \frac{cn \log\log n}{4130 \log n},$$

where $c = c' - 4$.

We are going to design Maker's strategy relying on the following theorem from Chapter 8 (see also [44]).

Theorem 5.8 *Let $12 \leq d \leq e^{\sqrt[3]{\log n}}$ and let $G = (V, E)$ be a graph on n vertices satisfying the following two properties:*

- *For every $S \subseteq V$, if $|S| \leq \frac{n \log\log n \log d}{d \log n \log\log\log n}$, then $|N(S)| \geq d|S|$;*
- *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \geq \frac{n \log\log n \log d}{4130 \log n \log\log\log n}$.*

Then G is Hamiltonian, for sufficiently large n.

We will also use the following lemmas, whose proofs appear in Subsection 5.3.1.

Lemma 5.9 *The random graph $G = G(n, p) = (V, E)$ satisfies the following properties a.s.:*

(P1) $\delta(G) \geq (\log\log n)^{c+2}$*;*

(P2) *Every vertex subset $A \subseteq V$ of cardinality $|A| \leq 4500k/(\log\log n)^c$ spans at most $|A|(\log\log n)^{c+1}$ edges in G;*

(P3) *For every two disjoint subsets A, B of sizes $|A| \leq 4500k/(\log\log n)^c$ and $|B| = |A|(\log\log n)^c$, the number of edges between A and B does not exceed $|A|(\log\log n)^{c+2}/600$;*

(P4) *For every two disjoint subsets A, B of V of cardinality $k/1000 \leq |A|, |B| \leq k$, we have $0.999|A||B|p \leq e(A, B) \leq 1.001|A||B|p$.*

Lemma 5.10 *Let $G = (V, E)$ be a graph on n vertices satisfying properties* **P1** *and* **P4***; then E can be split into two disjoint sets $E = E_1 \cup E_2$ such that, denoting $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, we have*

- $\delta(G_1) \geq (\log\log n)^{c+2}/100$*;*

- *For every two disjoint subsets A, B of V, such that $|A| = |B| = k$, we have*
$$e_{G_2}(A, B) \geq 0.1k^2 p \ .$$

Lemma 5.11 *Let H be a graph of minimum degree d; then in a $(1,1)$ Maker-Breaker game played on H, Maker can build a graph M with minimum degree at least $\left\lfloor \frac{\lfloor d/2 \rfloor}{2} \right\rfloor$.*

Lemma 5.12 *Let $G = (V, E)$, where $|V| = n$, be a graph that satisfies properties* **P2**, **P3**. *Let M_1 be a spanning subgraph of G of minimum degree $\delta(M_1) \geq \frac{(\log \log n)^{c+2}}{500}$. Then in M_1, every subset $A \subseteq V$ of cardinality $|A| \leq 4500k/(\log \log n)^c$, satisfies $|N_{M_1}(A)| \geq |A|(\log \log n)^c$.*

We are now ready to describe Maker's strategy. From Lemma 5.9 we have that $G(n, p)$ satisfies properties **P1** - **P4** a.s.; hence, from now on, we assume that the board is an edge-set of a graph G that satisfies all these properties. Before the game starts, Maker splits the board into two parts, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, as in Lemma 5.10. He plays two separate games in parallel, one on E_1 and the other on E_2. Whenever Breaker claims some edge of E_i, $i = 1, 2$, Maker plays in E_i as well (except for maybe once if Breaker has claimed the last edge of E_i). Let M denote the graph built by Maker and let $M_1 = M \cap E_1$, $M_2 = M \cap E_2$.

The game on E_1 is played according to Lemma 5.11 and so at the end of the game, Maker's graph M_1 will have minimum degree at least $\frac{(\log \log n)^{c+2}}{500}$. Since G satisfies properties **P2** and **P3**, it follows by Lemma 5.12 that every $A \subseteq V$ of cardinality at most $4500k/(\log \log n)^c$ satisfies $|N_{M_1}(A)| \geq |A|(\log \log n)^c$.

Playing on E_2, Maker's goal is to claim an edge between every two disjoint subsets $A, B \subseteq V$ of cardinality k. Using Theorem 5.6 on the hypergraph $\mathcal{H}$ whose set of vertices is E_2 and whose hyperedges are the edge sets of all bipartite subgraphs of G_2 with parts of size k we get

$$\begin{aligned}
\sum_{D \in \mathcal{H}} 2^{-|D|} &\leq \binom{n}{k}^2 2^{-0.1k^2 p} \\
&\leq \left(\frac{en}{k}\right)^{2k} e^{-(0.1 \log 2)k^2 p} \\
&\leq \exp\{k(2 \log \log n - (0.1 \log 2)kp)\} \\
&= o(1),
\end{aligned}$$

where the last equality follows since c is sufficiently large.

It follows that Maker can build a graph $M = M_1 \cup M_2$, for which

- Every subset $A \subseteq V$ of cardinality at most $4500k/(\log \log n)^c$ satisfies
$$|N_M(A)| \geq |A|(\log \log n)^c;$$

- There is an edge between every two disjoint subsets $A, B \subseteq V$ of cardinality $|A| = |B| = k$.

Thus, the Hamiltonicity of M follows from Theorem 5.8 by substituting $d = (\log\log n)^c$, for sufficiently large n. $\square$

5.3.1 Proofs of the lemmas

Proof of Lemma 5.9. Properties **P1 - P4** follow by standard first moment calculations and standard bounds on the tail of the binomial distribution.

P1: Let X be a random variable that counts the number of vertices of degree at most $(\log\log n)^{c+2}$. Then

$$\begin{aligned} E(X) &\le n \sum_{i=0}^{(\log\log n)^{c+2}} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\ &\le n \sum_{i=0}^{(\log\log n)^{c+2}} ((1+o(1))\log n)^i e^{-\log n} e^{-(\log\log n)^{c+4}} e^{2\log n (\log\log n)^{c+2}/n} \\ &\le e^{(1+o(1)(\log\log n)^{c+3} - (\log\log n)^{c+4}} \\ &= o(1). \end{aligned}$$

It follows by Markov's inequality that $Pr[X > 0] = o(1)$.

P2: Let $A \subseteq V$ be any subset of size $1 \le a \le 4500k/(\log\log n)^c$. Let X_A be a random variable that counts the number of edges with both endpoints in A. Then $X_A \sim Bin(\binom{a}{2}, p)$ and so $\mathbb{E}(X_A) = \binom{a}{2}p$. In order to bound the probability that X_A is much larger than its expectation, we use the following simplification of the original Chernoff bound.

Lemma 5.13 *[7, Lemma 2.1] If $X \sim Bin(n,p)$ and $k \ge np$ then $Pr(X \ge k) \le (enp/k)^k$.*

Then we have

$$\begin{aligned}
& Pr[\exists A \subseteq V \text{ with } 1 \leq a \leq 4500k/(\log\log n)^c \text{ and } X_A \geq a(\log\log n)^{c+1}] \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \binom{n}{a} Pr[X_A \geq a(\log\log n)^{c+1}] \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\frac{en}{a} \left(\frac{e\binom{a}{2}p}{a(\log\log n)^{c+1}} \right)^{(\log\log n)^{c+1}} \right]^a \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\exp\left\{1+\log(n/a) - (\log\log n)^{c+1}\left(\log(n/a) - \log\log n\right)\right\}\right]^a \\
= & \; o(1),
\end{aligned}$$

where the last equality follows from the upper bound on a.

P3: Let $A \subseteq V$ be any subset of cardinality $1 \leq a \leq 4500k/(\log\log n)^c$ and let B be any subset of $V \setminus A$ of cardinality $b = a(\log\log n)^c$. Let X_{AB} be a random variable that counts the number of edges with one endpoint in A and the other in B. Then $X_{AB} \sim Bin(ab, p)$ and so $\mathbb{E}(X_{AB}) = abp = a^2p(\log\log n)^c$. Let E denote the event *"there exist two disjoint subsets $A, B \subseteq V$, of sizes $1 \leq |A| = a \leq 4500k/(\log\log n)^c$ and $|B| = b = a(\log\log n)^c$, such that $e(A,B) > a(\log\log n)^{c+2}/600$"*. Using Lemma 5.13 we get

$$\begin{aligned}
Pr[E] \leq & \sum_{a=1}^{4500k/(\log\log n)^c} \binom{n}{a}\binom{n}{b} Pr[X_{AB} \geq a(\log\log n)^{c+2}/600] \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} \left[\frac{en}{a} \left(\frac{en}{b}\right)^{(\log\log n)^c} \left(\frac{eabp}{a(\log\log n)^{c+2}/600} \right)^{(\log\log n)^{c+2}/600} \right]^a \\
\leq & \sum_{a=1}^{4500k/(\log\log n)^c} [\exp\{1+\log(n/a) + (\log\log n)^c\left(1+\log(n/b)\right) \\
- & \frac{(\log\log n)^{c+2}}{600}\left(\log(n/b) + c\log\log\log n - \log\log n\right)\}]^a \\
= & \; o(1),
\end{aligned}$$

where the last equality follows from the upper bound on a and the definition of b.

P4: Let $A, B \subseteq V$ be any two disjoint subsets of sizes $k/1000 \leq a, b \leq k$ respectively. Let X_{AB} be a random variable that counts the number of edges with one endpoint in A and the other in B; then $X_{AB} \sim Bin(ab, p)$ and so $\mathbb{E}(X_{AB}) = abp$. Let E denote the event: "there exist two disjoint subsets $A, B \subseteq V$, of sizes $k/1000 \leq a, b \leq k$ respectively, such that $|e(A,B) - abp| > 0.001abp$". Using Chernoff's bound on the tail of the binomial distribution (that is, $Pr[|X - np| > \varepsilon np] \leq 2\exp\{-\frac{\varepsilon^2}{3}np\}$, where $X \sim Bin(n,p)$ and $0 < \varepsilon < 1$; see, e.g., [50, Corollary 2.3.]), we obtain

$$
\begin{aligned}
Pr[E] &\leq \sum_{a=k/1000}^{k} \sum_{b=k/1000}^{k} \binom{n}{a}\binom{n}{b} Pr[|X_{AB} - abp| > 0.001abp] \\
&\leq \sum_{a=k/1000}^{k} \sum_{b=k/1000}^{k} \left(\frac{en}{a}\right)^a \left(\frac{en}{b}\right)^b 2\exp\left\{-\frac{abp}{3\cdot 10^6}\right\} \\
&\leq k^2 \left(\exp\left\{2\log\log n - \frac{c}{3\cdot 4130\cdot 10^{12}}\log\log n\right\}\right)^k \\
&= o(1),
\end{aligned}
$$

where the last equality follows by choosing c to be sufficiently large. □

Proof of Lemma 5.10. Let each vertex $v \in V$ choose exactly $(\log\log n)^{c+2}/100$ of the edges incident with it, uniformly at random and independently of the choices for other vertices. We put an edge $e = (u, v)$ in E_1 iff for u or v (or both) e is chosen. Set $E_2 = E \setminus E_1$. Clearly, $\delta(G_1) \geq (\log\log n)^{c+2}/100$. We are going to use the following proposition, after proving it.

Proposition 5.14 *Let $G = (V, E)$ be a graph, satisfying property* **P4***; then, for every pair of disjoint subsets $A, B \subseteq V$ of cardinality $|A| = |B| = k$, there exist subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that*

- *$e(A_1, B_1) \geq 0.197k^2p$;*
- *Every $a \in A_1$ satisfies $0.4kp \leq d(a, B_1) \leq 1.1kp$;*
- *Every $b \in B_1$ satisfies $0.4kp \leq d(b, A_1) \leq 1.1kp$.*

Proof of Proposition 5.14. Let $A, B \subseteq V$ be any two disjoint subsets of cardinality k and let

$$
\begin{aligned}
X &= \{a \in A : d(a, B) \geq 1.1kp\}, \\
Y &= \{b \in B : d(b, A) \geq 1.1kp\}.
\end{aligned}
$$

Then we have $|X|, |Y| < k/1000$. Otherwise, due to property **P4**, we would have

$$|X| \cdot 1.1kp \leq e(X, B) \leq 1.001|X||B|p,$$

a contradiction; a similar argument applies to Y. Let $A_0 = A \setminus X$, $B_0 = B \setminus Y$, and let H_0 be the bipartite subgraph of G with bipartition (A_0, B_0). By the above, $0.999k \leq |A_0|, |B_0| \leq k$ and thus, by property **P4** we have $|E(H_0)| \geq 0.999|A_0||B_0|p \geq (0.999)^3k^2p \geq 0.997k^2p$.

Now, start with $H_1 = H_0$, and keep deleting vertices of degree at most $0.4kp$ until there are none left. Altogether we delete at most $0.4kp(|A_0|+|B_0|) \leq 0.8k^2p$ edges, so the resulting graph H_1 has at least $0.197k^2p$ edges and all degrees between $0.4kp$ and $1.1kp$; denote its parts by A_1 and B_1. This concludes the proof of the proposition. □

We return to the proof of the lemma. Let A, B be disjoint subsets of V of cardinality k, and let $A_1 \subseteq A$, $B_1 \subseteq B$, be the subsets whose existence is guaranteed by Proposition 5.14.

For every $a \in A_1$, denote by X_a the random variable, counting the number of edges from a to B_1, that were chosen by a. Then X_a, $a \in A_1$ are independent hypergeometric random variables with parameters $d_G(a)$, $d(a, B_1)$ and $(\log\log n)^{c+2}/100$ (that is, exactly $(\log\log n)^{c+2}/100$ elements of $\{(a,u) \in E : u \in V\}$ are chosen uniformly at random without replacement, and we count how many of them are in the set $\{(a,u) \in E : u \in B_1\}$). Hence

$$\mathbb{E}[X_a] = \frac{d(a,B_1)\frac{(\log\log n)^{c+2}}{100}}{d_G(a)} \leq \frac{d(a,B_1)}{100} < 0.02kp\,,$$

and thus, applying standard bounds on the tail of the hypergeometric distribution (see, e.g., [50], Theorem 2.10) we obtain

$$Pr[X_a > 0.03kp] < \exp\{-0.07 \cdot 0.03kp\} < \exp\left\{-\frac{kp}{600}\right\}.$$

Hence, the probability that there are at least $0.01k$ vertices in A_1, choosing at least $0.03kp$ edges leading to B_1 each, is at most

$$\binom{|A_1|}{0.01k} \exp\left\{-\frac{kp}{600} \cdot 0.01k\right\} \leq \exp\left\{-\frac{k^2p}{7 \cdot 10^4}\right\}.$$

Similarly, the probability that there are at least $0.01k$ vertices in B_1, choosing at least $0.03kp$ edges leading to A_1 each, is at most

$$\binom{|B_1|}{0.01k} \exp\left\{-\frac{kp}{600} \cdot 0.01k\right\} \leq \exp\left\{-\frac{k^2p}{7 \cdot 10^4}\right\}.$$

Thus, the probability that there exist subsets $A, B \subseteq V$ of cardinality k for which at least $0.01k$ vertices of A_1 choose at least $0.03kp$ edges leading to B_1 each, or at least $0.01k$ vertices of B_1 choose at least $0.03kp$ edges leading to A_1 each, is at most

$$\begin{aligned}
\binom{n}{k}^2 2\exp\left\{-\frac{k^2p}{7 \cdot 10^4}\right\} &\leq \left(\frac{en}{k}\right)^{2k} 2\exp\left\{-\frac{k^2p}{7 \cdot 10^4}\right\} \\
&\leq \left[\exp\left\{2\log\log n - \frac{kp}{7 \cdot 10^4}\right\}\right]^k \\
&= o(1),
\end{aligned}$$

where the last equality follows by choosing c to be sufficiently large.

It follows that there exists a choice of edges for the vertices, such that for every $A, B \subseteq V$ the total number of edges chosen between A_1 and B_1 is at most

$$0.01k \cdot 1.1kp + |A_1| \cdot 0.03kp + 0.01k \cdot 1.1kp + |B_1| \cdot 0.03kp < 0.09k^2p\ .$$

Hence

$$\begin{aligned} e_{G_2}(A,B) &\geq e_{G_2}(A_1,B_1) \\ &\geq e(A_1,B_1) - e_{G_1}(A_1,B_1) \\ &\geq 0.197k^2p - 0.09k^2p \\ &> 0.1k^2p. \end{aligned}$$

This concludes the proof of the lemma. □

Proof of Lemma 5.11. Let H^* be the graph, obtained from H by adding a new vertex v^* and connecting it to every vertex of odd degree in H (if all the degrees in H are even, then set $H^* = H$). Since all the degrees in H^* are even, it has an Eulerian orientation $\overrightarrow{H^*}$. For every $v \in V(H)$, let $E(v) = \{(v,u) \in E(H) : \overrightarrow{(v,u)} \in E(\overrightarrow{H^*})\}$. Clearly, $|E(v)| \geq \lfloor d_H(v)/2 \rfloor$ and the sets $E(v), v \in V(H)$ are pairwise disjoint. Using an obvious pairing strategy, Maker can claim at least $\lfloor |E(v)|/2 \rfloor$ edges from every set $E(v)$. □

Proof of Lemma 5.12. Let A be a subset of V of cardinality $1 \leq |A| \leq 4500k/(\log\log n)^c$. By property **P2** we have, $e_{M_1}(A) \leq e_G(A) \leq |A|(\log\log n)^{c+1}$, and thus there are at least $\delta(M_1)|A| - 2e_{M_1}(A) > \frac{|A|(\log\log n)^{c+2}}{600}$ edges leaving A in M_1. Thus $|N_{M_1}(A)| \geq |A|(\log\log n)^c$, as otherwise we get a contradiction with property **P3**. □

5.4 H-game one-on-one

Proof of Theorem 5.2. Instead of $G(n,p)$ we will work with the random graph model $G(n,M')$ where $M' = \Theta(p\binom{n}{2})$, the exact value of M' will be determined later. We will then use standard arguments from the theory of random graphs to pass from $G(n,M')$ to $G(n,p)$.

We adapt the analysis of Maker's strategy given by Bednarska and Łuczak [20] for biased games on the complete graph. Maker will make use of the following lemma, which was essentially proved in [20, Lemma 4].

Lemma 5.15 *Let H be a graph containing a cycle and let $M = 2\lfloor n^{2-1/m_2(H)} \rfloor$. There exists a real number $0 < \delta < 1$, such that a.s. every subgraph of the random graph $G(n,M)$ with $\lfloor (1-\delta)M \rfloor$ edges contains a copy of H.*

Since for every $M \leq M'$ a random subgraph of $G(n,M')$ with M edges is distributed according to $G(n,M)$, we obtain the following corollary:

Corollary 5.16 *Let H, M, δ be as in Lemma 5.15. Then for every $M' \geq M$, the random graph G generated according to $G(n, M')$ is almost surely such that almost every subgraph G_0 of G with $e(G_0) = M$ has the following property: every $(1-\delta)M$ edges of G_0 contain a copy of H.*

Assume that indeed $G \sim G(n, M')$ has the property stated in Corollary 5.16. Maker's strategy will be a random one. If we prove that Maker is able to win with probability $1-o(1)$, then we proved that Maker wins a.s. in $G(n, M')$. Indeed, if a random strategy of Maker succeeds with positive probability when playing on the edges of a particular fixed graph G on n vertices, then *there exists* a deterministic winning strategy for Maker in the game on G.

Maker's strategy is very simple. In each of his moves he chooses one of the edges of G that was not previously claimed by him, uniformly at random. If the edge is unclaimed, then Maker claims it and this move is considered to be *successful*. Otherwise, he claims an arbitrary unclaimed edge which will not be considered for future analysis; such a move is defined as *unsuccessful*.

Let $0 < \delta < 1$ be chosen so that the conditions of Lemma 5.15 are satisfied. We look at the board after $M = 2\lfloor n^{2-1/m_2(H)} \rfloor$ moves have been played by Maker.

Choose $M' = 4M/\delta$. Then, after the mth round, $m \leq M$, at most a $\delta/2$ fraction of the total number of elements of the board $E(G)$ is claimed (by both players). Therefore, for every $m \leq M$, the probability that the edge randomly chosen by Maker in his mth move was previously claimed by Breaker is bounded from above by $\delta/2$. It follows that a.s. at least $(1-\delta)M$ of Maker's first M moves are successful.

Obviously, the graph spanned by Maker's edges claimed during his first M moves (both successful and unsuccessful) can be viewed as the random subgraph of G with M edges. Applying Corollary 5.16, we get that the graph containing edges claimed by Maker in his successful moves contains a copy of H a.s., which means that Maker had won.

Finally we can argue that if $p \geq 1.01M'/\binom{n}{2}$, then a.s. in $G(n, p)$ Maker wins the H-game. This is due to the fact that the property of being a Maker's win is monotone (see, e.g., Chapter 2 of [25] for more details of this standard argument). $\square$

Proof of Theorem 5.3. Without loss of generality we can assume that H is 2-balanced, otherwise we can pass to a 2-balanced subgraph $H_1 \subseteq H$ such that $d_2(H_1) = m_2(H_1) = m_2(H)$ and provide Breaker with a strategy to prevent Maker from building a copy of H_1; this clearly entails his win in the H-game.

Let $(F_1, \ldots, F_s)$ be a sequence of copies of H. If $V(F_i) \setminus (\cup_{j=1}^{i-1} V(F_j)) \neq \emptyset$ and $|V(F_i) \cap (\cup_{j<i} V(F_j))| \geq 2$, for every $i = 2, \ldots, s$, then $F = \cup_{i=1}^{s} F_i$ is called an *s-bunch of H*, or simply a *bunch of H*.

Let L be an arbitrary graph and consider the auxiliary graph L_H with vertices corresponding to the subgraphs of L which are isomorphic to H, two vertices being adjacent iff the corresponding graphs have at least one edge in common. Let $F_1, \dots, F_s$ be the copies of H in L corresponding to some connected component of L_H. The graph $\cup_{i=1}^{s} F_i$ is called a *collection of* H. Collections are pairwise disjoint and they partition those edges of L which participate in a copy of H. It is clear that if the H-game is played on the edges of L, then Breaker has a winning strategy if and only if he can win the H-game on *every* collection of L.

Let us fix an $\varepsilon > 0$ in the definition of $p(n)$. We will show that

(i) Almost surely, every collection C of H in $G(n,p)$ has maximum density $m(C) < m_2(H) - \varepsilon$.

(ii) Playing on a graph C with $m(C) < m_2(H) - \varepsilon$, Breaker has a winning strategy in the H-game.

First, let us prove first the theorem provided that (i) and (ii) are true. By (i) we can assume that all collections of H in $G(n,p)$ have maximum density $m(C) < m_2(H) - \varepsilon$. Breaker will always select an edge from the same collection as the one Maker had selected from in the previous step. Within each of the collections, Breaker follows his winning strategy guaranteed by (ii). Hence he makes sure that he wins the H-game on each of the collections and thus wins the whole game on $G(n,p)$.

Let us now return to the proofs of (i) and (ii).

Proof of (i)**.** First we will show that there exists a constant $K = K(\varepsilon, H)$ such that almost surely every collection of H in $G(n,p)$ spans less than K vertices of $G(n,p)$.

In every collection C of H on c vertices there is also a bunch B of H spanning the same set of vertices. That bunch contains a sequence of at least $b = \lfloor c/v(H) \rfloor$ copies of H.

For a copy H' of H which is added to a bunch B', define $v_{\text{old}} = |V(H') \cap V(B')|$, $v_{\text{new}} = |V(H') \setminus V(B')|$, $e_{\text{old}} = |E(H') \cap E(B')|$, $e_{\text{new}} = |E(H') \setminus E(B')|$.

Under this notation we have:

$$d_2(H) = \frac{e(H')-1}{v(H')-2} = \frac{e_{\text{new}} + e_{\text{old}} - 1}{v_{\text{new}} + v_{\text{old}} - 2}.$$

If $v_{\text{old}} = 2$, then $\frac{e_{\text{new}}}{v_{\text{new}}} \geq d_2(H)$. Otherwise, $\frac{e_{\text{old}}-1}{v_{\text{old}}-2} \leq d_2(H)$ as H is 2-balanced and so again it follows that $\frac{e_{\text{new}}}{v_{\text{new}}} \geq d_2(H)$. We use this to show that the density of a bunch of large constant size can get arbitrary close to the 2-density of H. Imagine that a bunch B is rebuilt from its sequence of copies of H, by starting from a single copy and then adding one copy at a time, then we have

$$d(B) = \frac{e(H) + e_{\text{new}}^{(2)} + \cdots + e_{\text{new}}^{(b)}}{v(H) + v_{\text{new}}^{(2)} + \cdots + v_{\text{new}}^{(b)}} \geq d_2(H) - \varepsilon'(b),$$

where $\varepsilon'(b)$ is a positive function tending to 0 when $b \to \infty$ (recall that, by the definition of a bunch, we have $v_{\text{new}}^{(i)} \geq 1$ for every $2 \leq i \leq b$).

Let $b_0 = b_0(\varepsilon, H)$ be the smallest integer, such that $\varepsilon'(b_0) < \varepsilon$. Then the density of every b_0-bunch is strictly larger than $m_2(H) - \varepsilon$. There are constantly many b_0-bunches (the constant depends only on ε and H), so almost surely $G(n,p)$ contains no b_0-bunch. A b_0-bunch spans less than $K = K(\varepsilon, H) := v(H) \cdot b_0$ vertices, entailing that a.s. $G(n,p)$ does not contain collections of H on more than K vertices. (See, e.g., Chapter 4 of [25] for this well known result in random graph theory.)

Since there are at most constantly many graphs on K vertices, almost surely *no* subgraph of $G(n,p)$ on at most K vertices has density larger than $m_2(H) - \varepsilon$. This implies that a.s. the maximum density of every collection of H in $G(n,p)$ is less than $m_2(H) - \varepsilon$.

Proof of (ii)**.** Since for every subgraph $C' \subseteq C$, we have

$$\frac{e(C')}{v(C')} < m_2(H) - \varepsilon < dgn(H) - \frac{1}{2},$$

we know that $\delta(C') < 2dgn(H) - 1$. Hence C is $(2dgn(H) - 2)$-degenerate. This means that there is an ordering $w_1, \ldots, w_s$ of the vertices of C such that for every i, w_i has at most $2(dgn(H) - 2)$ edges going towards vertices of smaller index. Applying an obvious pairing strategy, Breaker can make sure that after the game is completed on C, for every i, w_i has at most $dgn(H) - 1$ edges of Maker going towards vertices of smaller index. This implies that Maker's graph on C is $(dgn(H) - 1)$-degenerate, and as such cannot contain a copy of H; it follows that Breaker had won. □

One class of graphs for which Theorems 5.2 and 5.3 do not hold is the class of trees, since a tree contains no cycles and $m_2(T) = 1 = dgn(T)$ for every tree T. The following observation allows us to investigate the tree game on the random graph locally.

Lemma 5.17 *Let T be a tree.*

(i) There exists a tree T' such that Maker can win the T-game $(E(T'), \mathcal{T}'_T, 1, 1)$, where $\mathcal{T}'_T$ is the set of all copies of T in T'.

(ii) Let $\overline{T}$ be a tree of minimal size such that Maker can win the T-game $(E(\overline{T}), \overline{\mathcal{T}_T}, 1, 1)$, where $\overline{\mathcal{T}_T}$ is the set of all copies of T in $\overline{T}$; then we have $p_{\mathcal{F}_T} = n^{-\frac{e(\overline{T})+1}{e(\overline{T})}}$.

Proof: (i) Assume that T is rooted arbitrarily, with depth ν. Let T' be the rooted tree of depth ν, such that for every $i = 0, 1, \ldots, \nu$ the down-degree of every vertex $v \in V(T')$ of the ith level is

$$\underline{d}(v) = 2 \cdot max_{\substack{u \in V(T) \\ u \text{ is on the } i\text{th level of } T}} \underline{d}(u),$$

that is, the down-degree of every vertex of the ith level of T' is double the maximum down-degree of the ith level of T. In the following, we present a winning strategy for Maker (as the second player) in the game $(E(T'), \mathcal{T}'_T, 1, 1)$, which will obviously entail the assertion of the lemma.

For every vertex $v \in V(T')$ the down-degree $\underline{d}(v)$ of v is even, and therefore one can pair up the edges going downwards from v. Maker applies an obvious pairing strategy, thus claiming half of the edges going down from every vertex of T'. At the end of the game, the graph claimed by Maker is a tree of depth ν, in which every vertex of the ith level, $0 \leq i \leq \nu$, has down-degree $max_{\substack{u \in V(T), \\ u \text{ on } i\text{th level of } T}} \underline{d}(u)$. Clearly, such a tree contains T as a subgraph, entailing Maker's win.

(ii) If $p = o\left(n^{-\frac{e(\overline{T})+1}{e(\overline{T})}}\right)$, then a.s. every connected component of $G(n,p)$ is a tree of order strictly less than $e(\overline{T})$. Since we assumed that $\overline{T}$ is a minimal tree on which Maker can win the T-game, he cannot win the game on any of the connected components of $G(n,p)$, and so clearly he cannot win the game on the whole edge-set of $G(n,p)$.

On the other hand, if $p = \omega\left(n^{-\frac{e(\overline{T})+1}{e(\overline{T})}}\right)$, then a.s. $G(n,p)$ contains a copy of $\overline{T}$. In this case, Maker can win the game on the edge-set of $G(n,p)$ by simply restricting his play to the edges of the aforementioned copy of $\overline{T}$. □

In the following three propositions, we use Lemma 5.17, in order to give some bounds on the threshold probability for some special classes of trees. In all three propositions we assume that Maker starts the game; of course similar results can be obtained if Breaker starts the game.

Proposition 5.18 *The threshold probability for the l-path game is*

$$p_{\mathcal{F}_{P_l}} = n^{-\frac{e_l+1}{e_l}}, \text{ where } e_l = \Theta\left(2^{l/2}\right).$$

Proof: According to Lemma 5.17, we need only show that if a minimal tree $\overline{T}$ on which Maker can win the $(E(\overline{T}), \overline{P_l}, 1, 1)$ game has e_l edges, then $e_l = \Theta\left(2^{l/2}\right)$.

It is easy to check that Maker can win the game on the tree T_0 that has a root of degree 3, and each of its sons is the root of a complete binary tree of depth $\lceil l/2 \rceil - 1$. Indeed, if he starts by claiming an edge incident with the root, and then proceeds by the pairing strategy described in the proof of Lemma 5.17 (i), he will claim two edges incident with the root, and one edge going down from every other vertex (which is not a leaf). Therefore, at the end of the game there will be two disjoint paths of length $\lceil l/2 \rceil$ each, incident with the root that are claimed by Maker, and thus also an l-path claimed by Maker. The size of the tree T_0 is $1 + 3 \cdot \left(2^{\lceil l/2 \rceil} - 2\right)$.

Next, we prove that Breaker can win the l-path game on the edges of any tree with at most $2^{l/2} - 1$ vertices. Observe that every path in a tree is uniquely determined by its endpoints, and therefore any tree T with at most $2^{l/2} - 1$ vertices has at most $\binom{v(T)}{2} < 2^{l-1}$ paths of length l. Applying Theorem 5.6 to the hypergraph of paths of length l in T, entails our claim. □

Proposition 5.19 *The threshold probability for the d-star game is*

$$p_{\mathcal{F}_{S_d}} = n^{-\frac{2d}{2d-1}}.$$

Proof: Regardless of his strategy, Maker wins the d-star game, played on the edges of a star with $2d-1$ edges.

On the other hand, Regardless of his strategy, Breaker wins the d-star game on any tree with less than $2d-1$ edges, since throughout the game, Maker will claim at most $d-1$ edges.

The proposition now follows by Lemma 5.17. □

As we saw in the last two propositions, the size of the smallest tree on which Maker can win the game is linear in terms of the size of the winning set for the star game, but exponential for the path game. Even though we cannot determine the threshold for an arbitrary tree, in the next proposition we analyze another, more general class of trees.

Proposition 5.20 *Let F be a rooted tree with depth l and let d_i, $i = 0, \ldots, l-1$ be integers such that every vertex $v \in V(F)$ on the ith level of F has down-degree d_i. Then, the threshold probability for the F-game is $p_{\mathcal{F}_F} = n^{-\frac{t+1}{t}}$, where*

$$2^{l-1/2} \cdot \sqrt{d_0(d_0-1)} \cdot d_1 \cdots d_{l-1} \leq t \leq \sum_{i=1}^{l} 2^i \cdot d_0 \cdot d_1 \cdots d_{i-1}.$$

Proof: As in the proof of Lemma 5.17 (i), Maker can win the game on a tree F_2 obtained from F by "doubling" down-degrees on every level, and then using a pairing strategy. The tree F_2 has $\sum_{i=1}^{l}(2d_0) \cdot (2d_1) \cdots (2d_{i-1}) = \sum_{i=1}^{l} 2^i \cdot d_0 \cdot d_1 \cdots d_{i-1}$ edges.

On the other hand, observe that the target tree F has $\binom{d_0}{2}\left(\prod_{i=1}^{l-1} d_i\right)^2$ paths of length $2l$ (to construct such a path choose a pair of sons of the root in $\binom{d_0}{2}$ ways, and then continue from each of the chosen vertices down to level l – there are $\prod_{i=1}^{l-1} d_i$ ways of doing it). Therefore, in order for Breaker to win the F-game on a tree T, it is enough to prevent Maker from owning $\binom{d_0}{2}\left(\prod_{i=1}^{l-1} d_i\right)^2$ paths of length $2l$. A tree T has at most $\binom{|V(T)|}{2}$ paths of length $2l$. Hence, by Theorem 5.7 Breaker has a winning strategy for the F-game on T if

$$\binom{v(T)}{2} \cdot 2^{-2l} < \frac{1}{2}\binom{d_0}{2}\left(\prod_{i=1}^{l-1} d_i\right)^2,$$

or $|V(T)| < 2^{l-1/2} \cdot \sqrt{d_0(d_0-1)} \cdot d_1 \cdots d_{l-1}$, as claimed. □

Note that both stars and paths of even length satisfy the conditions of Proposition 5.20. For $(2l)$-paths we have

$$2^{l-1} \leq t \leq \sum_{i=1}^{l} 2^i \cdot 2 \leq 2^{l+2},$$

and for d-stars we get $d-1 \leq t \leq 2d$. Hence, even though it is not as precise as the previous two propositions, Proposition 5.20 still gives the correct order of magnitude.

5.5 Concluding remarks and open problems

Min-degree-game. We think that the problem considered in Lemma 5.11 is very interesting in its own right. It was proved that in any graph with minimum degree d, Maker can create a graph with minimum degree $\lfloor d/4 \rfloor$. For $d \gg \log n$ it is known (see, e.g. [17]) that Maker can create a graph with minimum degree $d/2 - o(d)$. It would be very interesting to decide whether this is true for constant d. Let m_d be the largest integer such that in every graph with minimum degree d Maker can create a graph with minimum degree at least m_d. According to Lemma 5.11 we have $m_d \geq \lfloor d/4 \rfloor$.

Problem 5.21 *Determine m_d asymptotically.*

***H*-game.** It follows from Theorems 5.2 and 5.3 that for every graph H that satisfies $m_2(H) \leq dgn(G) - \frac{1}{2}$ we have $n^{-m_2(H)^{-1}-\varepsilon} \leq p_{\mathcal{F}_H} \leq n^{-m_2(H)^{-1}}$, for any $\varepsilon > 0$. On the other hand we saw examples of graphs, like K_3 and any tree $T \neq K_2$, for which the assertion of Theorem 5.3 does not hold. It would be very interesting to characterize these graphs. In particular, could the condition of Theorem 5.3 be made into an if and only if statement?

Problem 5.22 *Let H be a graph such that $m_2(H) > dgn(H) - \frac{1}{2}$. Prove that there is an $\varepsilon = \varepsilon(H) > 0$ such that a.s. Maker has a winning strategy for the H-game on $G(n,p)$, provided $p = n^{-\frac{1}{m_2(H)-\varepsilon}}$.*

Hamiltonicity game. In [70] it was conjectured that, provided that Breaker starts the game, essentially the only reason for Maker to lose the Hamiltonicity game on $G(n,p)$ is that the graph has a vertex of degree 3. One can formulate this conjecture precisely in the model of the random graph process: the hitting time of the event that Maker has a winning strategy in the Hamiltonicity game is *equal* to the hitting time of the event that the minimum degree of the graph is 4 (provided that Breaker starts the game). A less ambitious goal would be to prove that the threshold probabilities coincide asymptotically.

Conjecture 5.23 *Suppose Breaker starts the game. If $p \geq \frac{\log n + 3\log\log n + \omega(1)}{n}$, where the $\omega(1)$ term tends to infinity with n arbitrarily slowly, then a.s. Maker wins the $(1,1)$ Hamiltonicity game on $G(n,p)$.*

If true, then this statement could be viewed as a game theoretic analog of the famous theorem of Komlós and Szemerédi (see [53]) on the Hamiltonicity of the random graph.

Chapter 6

Games on sparse graphs

6.1 Introduction

Let p and q be two positive integers, X a finite set, and $\mathcal{F} \subseteq 2^X$ a hypergraph. In the positional game $(X, \mathcal{F}, p, q)$, two players take turns claiming previously unclaimed elements of X. In every move, the first player claims p elements, and then the second player responds by claiming q elements. The set X is called the "board", and p and q are the biases of the first and second player, respectively. For the purposes of this chapter $\mathcal{F}$ is assumed to be monotone increasing. In a *Maker/Breaker-type* positional game, the two players are called Maker and Breaker and $\mathcal{F}$ is referred to as the family of winning sets. Maker wins the game if the subset of X he claims by the end of the game (that is, when every element of the board has been claimed by one of the players) is a winning set, that is, an element of $\mathcal{F}$; otherwise Breaker wins. Since $\mathcal{F}$ is monotone increasing, Maker wins if and only if he occupies an inclusion-minimal element of $\mathcal{F}$. Whenever there is no risk of confusion we use $\mathcal{F}$ to denote also the family of inclusion-minimal members of $\mathcal{F}$.

The study of positional games on the set of edges of a (complete) graph was initiated by Lehman [60] who, in particular, proved that Maker can easily win the $(E(K_n), \mathcal{T}_n, 1, 1)$ game, where the family $\mathcal{T}_n$ consists of the edge-sets of all connected and spanning subgraphs of K_n (by "easily" we mean that he can do so within $n-1$ moves, which is clearly optimal). Chvátal and Erdős [30] suggested to "even out the odds" by giving Breaker more power, that is, by increasing his bias. They determined that the connectivity game $(E(K_n), \mathcal{T}_n, 1, b)$ is won by Maker even when the bias b of Breaker is as large as $cn/\log n$, for some small constant $c > 0$, whereas for another constant $C > 0$, Breaker wins the game if his bias is at least $Cn/\log n$. They also showed that the $(E(K_n), \mathcal{H}_n, 1, 1)$ Hamiltonicity game, in which Maker's goal is to build a Hamiltonian cycle (that is, the family $\mathcal{H}_n$ of winning sets consists of the edge-sets of all Hamiltonian subgraphs of K_n), is won by Maker for sufficiently large n. Moreover, they conjectured that in fact Maker can win the $(E(K_n), \mathcal{H}_n, 1, b)$ game for some b that tends to infinity with n. This was proved by Bollobás and Papaioannou [27], who showed that Maker wins Hamiltonicity against a bias of $O(\log n/\log\log n)$. Finally, Beck [9] gave a winning strategy for Maker against a bias of $O(n/\log n)$. Very recently, Krivelevich and Szabó [58] were able to improve Beck's bound (by a constant factor).

In this chapter, we are looking for the sparsest graph on which Maker can win a certain game; the definition of "sparseness" may vary for different games. For example, by Lehman's Theorem we know that, for every integer $n \geq 4$, **there exists** a graph with $2n-3$ edges ($2n-2$ if Breaker is the first player) on which Maker can win the $(1,1)$ connectivity game. On the other hand, Breaker can win this game on **every** graph on n vertices and at most $2n-4$ edges ($2n-3$ if Breaker is the first player). Clearly we cannot expect Maker to win on **every** graph with a "large" average degree as such a graph might be disconnected. If we change the definition of sparseness we consider, and take the board to be a highly connected graph, then by Lehman's Theorem, and the famous theorems of Tutte [75] and Nash-Williams [65], it follows that Maker (as first or second player) can win the $(1,1)$ connectivity game on every 4-connected graph, whereas there exists a 3-connected graph on which Breaker wins.

Coming back to ensuring a large average degree, we note that Lehman's Theorem does not hold for the biased game $(1,q)$, $q>1$. In fact, for every r there exist graphs with minimum degree r (or r-edge-connected graphs, or graphs that admit r pairwise edge disjoint spanning trees) on which Breaker wins the $(1,1)$ connectivity game (see [30, 42]). On the other hand, it is known [6, 30] that for every $\varepsilon > 0$, Maker wins the $(1,(\log 2-\varepsilon)n/\log n)$ connectivity game, played on the edges of K_n, but loses the $(1,(1+\varepsilon)n/\log n)$ game.

For the Maker-Breaker $(1,k)$ connectivity game, played on the edges of some graph G, let $c(G,k) = e(G)/(k+1)$ (more precisely $c(G,k)$ in this case is the number of edges that Maker will have in the end of the game - when every element of the board is claimed by some player) if Maker wins the game and $c(G,k) = \infty$ otherwise. Let $C_n(k) = \min c(G,k)$, where the minimum is extended over all graphs on n vertices.

In this notation, Lehman's Theorem implies that $C_n(1) = n-1$ for every $n \geq 4$. Moreover, the aforementioned results of Chvátal and Erdős [30] and of Beck [6] imply that $C_n((\log 2 - \varepsilon)n/\log n) = O(n \log n)$. We prove that $C_n(k) = \Theta(n \log n)$ for every $2 \leq k \leq (\log 2 - \varepsilon)n/\log n$.

In Section 6.2 we prove the following more general result regarding the biased connectivity game:

Theorem 6.1 *1. For every $\varepsilon > 0$, integer $q \geq 2$, sufficiently large even n and $d \geq (1+\varepsilon)q\log_2 n$ there exists a graph G on n vertices with average degree $(1+o(1))d$, such that Maker wins the $(1,q)$ connectivity game, played on the edges of G.*

2. Let $G=(V,E)$ be any d-regular graph on n vertices and let $\varepsilon > 0$. If $d < (1/2-\varepsilon)(q-1)\log n$, then Breaker has a winning strategy for the $(1,q)$ connectivity game on G.

Let $G=(V,E)$ be a graph and let $\mathcal{H}_G$ be the hypergraph whose vertices are the edges of G and whose (inclusion-minimal) hyperedges are the edge-sets of all Hamilton cycles of G. We are interested in the sparsest (in terms of the average degree) graph G on which Maker has a winning strategy in the $(1,1,\mathcal{H}_G)$ game. By Theorem 5.1 we know that an average degree of $\bar{d} = (1+o(1))\log n$ is sufficient. On the other hand, if there are at least three vertices of G of degree at most 3 (sometimes even fewer than three vertices suffice), then Breaker can easily win the game; hence $\bar{d} \geq 4 - o(1)$ is necessary. We prove that nevertheless a (somewhat larger) constant degree is sufficient to entail Maker's win.

Theorem 6.2 *There exist infinitely many values of n for which there exists a graph $G = (V, E)$ on n vertices with average degree $\bar{d} \leq 1003$; such that Maker has a winning strategy for the $(1, 1, \mathcal{H}_G)$ game of "Hamiltonicity" played on E.*

A graph $G = (V, E)$ is called an *expander* if every "not too large" subset $S \subset V$ has a "large" neighborhood in G. Expanders are known to have many "good" graph theoretic properties; thus building an expander is a very useful goal for Maker. Clearly, in order for Maker to have a chance, the board itself has to be an expander.

A graph G is said to be (n, d, λ) if G is a d-regular graph on n vertices and its second largest (in absolute value) eigenvalue is λ. It is well known (see e.g. [3]) that if $\lambda \ll d$, then G is a "good" expander. We prove, that playing on the edges of an (n, d, λ)-graph, Maker can build an expander.

Theorem 6.3 *Playing a $(1, 1)$ Maker-Breaker game on the edges of an (n, d, λ)-graph $G = (V, E')$ with $\lambda = o(d)$, Maker can build a subgraph $H = (V, E)$ in which $|N(A)| \geq \frac{d^2}{300\lambda^2}|A|$ for every $A \subset V$ of size at most $\lambda^2 n/d^2$, and $|N(A)| \geq n/81 - |A|$ for every $A \subset V$ of size at least $\lambda^2 n/d^2$.*

In the Maker-Breaker planarity game, played on the edge-set of some graph G, Maker's goal is to claim the edges of some non-planar subgraph of G. For (almost) every positive integer q, we determine the minimal "sparseness" needed to ensure Maker's win in the $(1, q)$ planarity game.

Theorem 6.4

1. *If $q \geq d/2$ then Breaker wins the $(1, q)$ planarity game played on any graph G on n vertices with maximum degree $\leq d$.*
2. (a) *For every positive integer d there exist infinitely many values of n such that if $q < d/2 - 1$, then there exists a d-regular graph G on n vertices such that Maker wins the $(1, q)$ planarity game, played on G.*

   (b) *For every $\varepsilon > 0$ there exists an integer $d_0 = d_0(\varepsilon)$ such that for every $d \geq d_0$ and every $q \leq (1/2 - \varepsilon)d$, Maker has a winning strategy for the $(1, q)$ planarity game on any d-regular graph G.*

   (c) *If $q \leq d/6 - 1$ then Maker wins the $(1, q)$ planarity game played on any graph G on n vertices with average degree $\geq d$.*

In the Maker-Breaker k-coloring game, played on the edge-set of some graph G, Maker's goal is to claim the edges of some non-k-colorable subgraph of G.

For every positive integer q we find "almost tight" bounds on the regularity of the graph which turns Breaker's win to Maker's win.

Theorem 6.5 *Let $k \geq 2$ be an integer.*

1. *For every positive integer q there exist infinitely many values of d such that if $q \leq \frac{d+1}{3k \log k}$ then there exists a d-regular graph G such that Maker wins the $(1, q)$ k-coloring game, played on G.*

2. *If $q \geq \frac{d}{2\lfloor k/2 \rfloor}$ then Breaker wins the $(1, q)$ k-coloring game played on any d-regular graph G on n vertices.*

We turn to a different setting where the board is an r-chromatic graph and Maker's goal is to build a graph with the highest chromatic number possible.

Theorem 6.6 1. *Let G be an arbitrary r-chromatic graph; then Maker can win the $(1, 1)$ $\sqrt{r}$-coloring game on G as the first player and the $(1, 1)$ $\sqrt{r-1}$-coloring game on G as the second player.*

2. *For every positive integers r and q and sufficiently large $n = n(r, q)$, there exists an r-chromatic graph G on n vertices, such that, playing the $(1, q)$ game on G, Maker can build an r-chromatic graph.*

In the Maker-Breaker $(1, q, t)$ minor game, played on the edge-set of some graph G, Maker's goal is to claim the edges of some K_t minor of G.

For every positive integer q we find tight bounds on the average degree, needed in order to turn Breaker's win to Maker's win.

Theorem 6.7 *Let $t \geq 3$ be an integer.*

1. *For every $\varepsilon > 0$ there exist infinitely many values of $n = n(\varepsilon, t)$ such that if $q \leq \frac{d}{2+\varepsilon} - 1$ then there exists a graph G on n vertices with average degree d, such that Maker wins the $(1, q, t)$ minor game, played on G.*

2. *If $q \geq d/2$ then Breaker wins the $(1, q, t)$ minor game played on any graph G on n vertices with maximum degree $\leq d$.*

Another interesting angle is the following. Maker and Breaker play a $(1, 1)$ game on the edges of some graph G that admits a K_t-minor. Maker wins the game iff the graph he builds admits a K_r-minor for some $r = r(t)$. For example G can be a graph which can only be embedded on a surface with a large genus, and Maker's goal will be to build a non-planar graph. We prove the following:

Theorem 6.8 *For infinitely many values of n there exists a graph G_n on n vertices that admits a $K_{\Theta(\sqrt{n})}$-minor such that, playing the $(1, 1)$ game on the edges of G_n, Breaker can force Maker to build a K_3-minor free graph.*

The rest of this chapter is organized as follows: in Section 6.2 we prove Theorem 6.1, in Section 6.3 we prove Theorem 6.2, in Section 6.4 we prove Theorem 6.3.

6.2 The connectivity game

Proof of Theorem 6.1:

1. Fix $\varepsilon > 0$ and let $G = (V, E)$ be a graph on n vertices with average degree at most $(1 + o(1))d$, in which
$$e(A, V \setminus A) \geq (1 - \varepsilon/2)d|A||V \setminus A|/n \text{ for every } A \subseteq V. \tag{6.1}$$
If $d \geq c\sqrt{n}$ for some $c > 0$, then a.s. $G \in G(n, d/n)$ satisfies (6.1); this follows from standard bounds on the tail of the binomial distribution. For smaller values of d, we prove the following:

Claim 6.9 *Let $\varepsilon > 0$ and let $G = (V, E)$ be a random n-vertex d-regular graph, where $\log n \leq d = o(\sqrt{n})$, then a.s. $e(A, V \setminus A) \geq (1 - \varepsilon/2)d|A||V \setminus A|/n$ for every $A \subseteq V$.*

Proof of Claim 6.9
In the proof we will use the following two results from [19] regarding the distribution of edges in $\mathcal{G}_{n,d}$.

Theorem 6.10 *If $d = o(\sqrt{n})$, then a.s. every subset U of the vertices of a graph, drawn uniformly at random from $\mathcal{G}_{n,d}$ satisfies*
$$\left| e(U) - \binom{|U|}{2}\frac{d}{n} \right| = O(|U|\sqrt{d}).$$

Theorem 6.11 *If $d = o(\sqrt{n})$, then a.s. every pair of disjoint subsets U, W of the vertices of a graph, drawn uniformly at random from $\mathcal{G}_{n,d}$ satisfies*
$$\left| e(U, W) - \frac{|U||W|d}{n} \right| = O(\sqrt{|U||W|d}).$$

Let $A \subseteq V$ be any subset of size $1 \leq a \leq n/\log\log n$. According to Theorem 6.10, the number of edges with both endpoints in A is a.s. at most $\frac{da^2}{2n} + O(a\sqrt{d}) = o(da)$. Since G is d-regular it follows that $e(A, V \setminus A) \geq (1 - o(1))da(n - a)/n$. Next, let $A \subseteq V$ be any subset of size $n/\log\log n \leq a \leq n/2$. According to Theorem 6.11, the number of edges with one endpoint in A and the other in $V \setminus A$ is a.s. at least $\frac{da(n-a)}{n} - O(\sqrt{da(n-a)}) = (1 - o(1))da(n - a)/n$. $\square$

In order to build a spanning connected graph (and thus win) Maker will claim an edge in every cut of G; that is, he will assume the role of Cut-Breaker in the cut game on G. Let $\mathcal{H}_G$ be the hypergraph whose vertices are the edges of G and whose hyperedges are the edge sets of all bipartite spanning induced subgraphs of G.

$$\begin{aligned}
\sum_{A\in\mathcal{H}} 2^{-|A|/q} &\leq \sum_{r=1}^{n/2} \binom{n}{r} 2^{-(1-\varepsilon/2)dr(n-r)/(qn)} \\
&\leq \sum_{r=1}^{n/2} \left[\frac{en}{r} 2^{-(1-\varepsilon/2)(1+\varepsilon)(1-r/n)\log_2 n}\right]^r \\
&\leq \sum_{r=1}^{\sqrt{n}} \left[en^{1-(1+\varepsilon')}\right]^r + \sum_{r=\sqrt{n}}^{n/2} \left[en^{1/2-(1/2+\varepsilon'')}\right]^r \\
&= o(1).
\end{aligned}$$

Applying Beck's criterion [6] we conclude that Cut-Breaker has a winning strategy for the $(q, 1)$ cut game on G and so Maker has a winning strategy for the $(1, q)$ connectivity game on G.

2. Let $\mathcal{H} = \{A_1, A_2, \ldots, A_k\}$ be a hypergraph where the A_i's are pairwise disjoint and $||A_i| - |A_j|| \leq 1$ for every $1 \leq i, j \leq k$. The $(1, q, \mathcal{H})$ Box Game (introduced in [30]), is played by two players (called Box-Maker and Box-Breaker), who take turns claiming elements of the board $\bigcup_{i=1}^k A_i$. Box-Breaker claims one element per move whereas Box-Maker claims q. Box-Maker wins the game iff he claims all the elements of A_i for some $1 \leq i \leq k$. It was proved in [30] that Box-Maker (as the second player) wins this game iff $\sum_{i=1}^k |A_i| \leq f(k, q)$, where $f(k, q)$ is some function satisfying

$$(q-1)k \sum_{i=1}^{k-1} 1/i \leq f(k,q) \leq qk \sum_{i=1}^{k-1} 1/i.$$

Assume first that $q \leq n^{\varepsilon/2}$. Let $S = \{v_1, v_2, \ldots, v_s\} \subseteq V$ be an independent set in G of size $s = n/d$. For every $1 \leq i \leq s$, let A_i denote the set of edges of G which are incident with v_i; note that $|A_i| = d$ for every $1 \leq i \leq s$. Breaker will simply play the Box Game (assuming the role of Box-Maker) on the hypergraph $\{A_1, A_2, \ldots, A_s\}$, and win as

$$\begin{aligned}
\sum_{i=1}^{s} |A_i| &\leq ds < s(1/2-\varepsilon)(q-1)\log n < s(1-2\varepsilon)(q-1)\log s \\
&< s(q-1)\sum_{i=1}^{s-1} 1/i \leq f(s,q)
\end{aligned}$$

Next, assume that $\sqrt{n} < q = o(n)$ (clearly if $q = cn$ for some constant $c > 0$, then Breaker wins the $(1, q)$ connectivity game on K_n). In this case Breaker's strategy is divided into two stages. In the first stage, Breaker claims all the edges of G with both endpoints in some subset A of V such that $|A| = s := q/2$ and all the vertices of A are isolated in Maker's graph. Breaker can do this within s rounds. Indeed, assume that after t rounds $(1 \leq t < s)$ Breaker has claimed all the edges with both endpoints in some subset A_t of V of size t where every vertex of A_t is isolated in Maker's graph. In the $(t+1)$st round Breaker arbitrarily chooses two vertices $u, v \in V \setminus A_t$ that are isolated in Maker's graph (this is possible as $q = o(n)$). If $(u, v) \in E$ then Breaker claims it. Moreover he claims all edges in $\{(u, w) : w \in A_t\} \cup \{(v, w) : w \in A_t\}$. This is possible as $t < s$. Let $A'_{t+1} = A_t \cup \{u, v\}$. On his next move, Maker claims some edge (x, y). Clearly $|A'_{t+1} \cap \{x, y\}| \leq 1$. Deleting an appropriate vertex from A'_{t+1} we obtain a set $A_{t+1} \subseteq V$ of size $t+1$ such that every vertex of A_{t+1} is isolated in Maker's graph. Denote $A_s = \{u_1, u_2, \ldots, u_s\}$.

In the second stage, Breaker will isolate u_i for some $1 \leq i \leq s$. Let $\mathcal{H}'_G = \{B_1, B_2, \ldots, B_s\}$ where, for $1 \leq i \leq s$, $B_i = \{(u_i, v) \in E : v \in V\}$. Clearly the B_i's are pairwise disjoint and $|B_i| \leq d$ for every $1 \leq i \leq s$. By adding some "fake" elements to every B_i which is strictly smaller than d we can obtain a hypergraph $\mathcal{H}_G = \{C_1, C_2, \ldots, C_s\}$ such that the C_i's are pairwise disjoint and $|C_i| = d$ for every $1 \leq i \leq s$. Breaker plays the Box Game (assuming the role of Box-Maker) on the hypergraph $\mathcal{H}_G$, and wins as

$$\begin{aligned}\sum_{i=1}^{s} |C_i| &\leq ds < s(1/2 - \varepsilon)(q-1)\log n < s(1-2\varepsilon)(q-1)\log s \\ &< s(q-1)\sum_{i=1}^{s-1} 1/i \leq f(s, q)\end{aligned}$$

□

6.3 The Hamilton cycle game

Proof of Theorem 6.2

Let $m \geq 2$ be an integer and let $G_n = (V, E)$ be a graph on $n = m(d+1)$ vertices where $d = 1001$. The graph G_n consists of m cliques $K_0, \ldots, K_{m-1}$, each of size d, and m additional vertices $u_0, \ldots, u_{m-1}$. For every $0 \leq i \leq m-1$ there is an edge in G_n between u_i and every vertex of K_i and of K_{i+1} (the indices are taken modulo m). Clearly the average degree of G_n is $\frac{2dm+md(d+1)}{m(d+1)} < d+2 = 1003$. The induced subgraph of G_n on $\{u_{i-1}, u_i\} \cup V(K_i)$ (modulo m) will be called the *ith part of* G_n.

Before describing Maker's strategy, we state the following useful lemma:

Lemma 6.12 *Maker can win the* $(1, 1)$ *Hamiltonicity game (as the first or second player), played on the edges of* K_n*, provided* $n > 1000$.

The assertion of this lemma follows immediately from the proof of Theorem 4.2.

Now we provide Maker with a winning strategy for the $(1, 1)$ Hamiltonicity game on G_n. Maker plays m separate games in parallel, that is, whenever Breaker claims some edge of the ith part of G_n, Maker claims an edge in the same part (this is always possible, except for maybe once for each part, if Breaker claims the last edge of this part; whenever this happens Maker claims an arbitrary edge). Thus, it suffices to prove that, playing as the first or second player on the first part of G_n, Maker can build a spanning path between u_0 and u_1. Again Maker divides the game into two separate games played in parallel, one played on the edges of K_1 and the other on the board $\hat{E} := \{(u_0, v) : v \in V(K_1)\} \cup \{(u_1, v) : v \in V(K_1)\}$. Playing on $E(K_1)$ Maker will build a spanning cycle (his win is guaranteed by Lemma 6.12). Playing on $\hat{E}$, Maker uses a simple pairing strategy; let $V(K_1) = \{v_0, v_1, \ldots, v_{1000}\}$. Maker pairs (u_0, v_0) with (u_1, v_0). Moreover, for every $i \in \{0, 1\}$ and every $1 \leq j \leq 500$, Maker pairs (u_i, v_{2j-1}) with (u_i, v_{2j}).

It remains to prove that, if Maker plays according to this strategy, then indeed he obtains a spanning path between u_0 and u_1. According to Maker's strategy and to Lemma 6.12, Maker's graph contains a spanning cycle $C = (v_0, v_1, \ldots, v_{d-1}, v_0)$ in K_1. Moreover, without loss of generality, we can assume that $deg_M(u_0) = t$ and $deg_M(u_1) = t + 1$, where $deg_M(v)$ denotes the degree of v in Maker's graph. Let $X \subseteq V(K_1)$ denote the set of neighbors of u_0 in Maker's graph. Let $L_C(X) := \{v_{i-1} : v_i \in X\}$ and $R_C(X) := \{v_{i+1} : v_i \in X\}$, where all indices are taken modulo d, denote the sets of "left" and "right" neighbors of X on C respectively; clearly $|L_C(X)| = |R_C(X)| = t$. Moreover, $L_C(X) \neq R_C(X)$; indeed, since $0 < |X| < |V(K_1)|/2$, there must exist a vertex $v_i \in X$ and $k \geq 2$ consecutive vertices $v_{i+1}, \ldots, v_{i+k} \in V(K_1) \setminus C$. Clearly $v_{i+1} \in R_C(X) \setminus L_C(X)$. Hence, $|L_C(X) \cup R_C(X)| \geq t+1$ and thus contains a neighbor of u_1. This concludes the proof of the theorem. □

6.4 Building expanders

In the proof of Theorem 6.3 we will use a (special case of a) result of Frieze and Molloy [39]. Before stating this result, we need the following definition.

Definition 6.13 *For a graph $G = (V, E)$ on n vertices let*

$$\Phi(G) = \min_{\substack{A \subseteq V \\ |A| \leq n/2}} \frac{e_G(A, V \setminus A)}{|A|}$$

denote the edge expansion of G.

We can now state their result:

Theorem 6.14 *Let $\varepsilon > 0$ be a real number and let $G = (V, E)$ be a d-regular graph with edge expansion Φ. If $\frac{d}{\log d} \geq 14\varepsilon^{-2}$ and $\Phi \geq 8\varepsilon^{-2} \log d$, then there exists a partition $E = E_1 \cup E_2$ such that, for $i = 1, 2$, the graph $G_i = (V, E_i)$ has edge expansion $\Phi_i \geq (1 - \varepsilon)\Phi/2$ and $(1 - \varepsilon)d/2 \leq \delta(G_i) \leq \Delta(G_i) \leq (1 + \varepsilon)d/2$.*

Proof of Theorem 6.3:
Being an (n, d, λ)-graph, $G = (V, E)$ satisfies the following properties:

(1) For every $A \subseteq V$, there are at most $\frac{d|A|^2}{2n} + \lambda|A|$ edges of G with both endpoints in A.

(2) For every $A \subseteq V$, there are at least $\frac{d|A|(n-|A|)}{n} - \lambda|A|$ edges of G with exactly one endpoint in A.

(3) For every two disjoint subsets $A, B \subseteq V$ there are at most $\frac{d|A||B|}{n} + \lambda\sqrt{|A||B|}$ edges with one endpoint in A and the other in B.

See e.g. [3] for a proof.

Note that, in particular, G satisfies the conditions of Theorem 6.14 for an appropriate choice of ε, with $\Phi = d/2 - \lambda$.

Hence, before the game starts, Maker partitions the board $G = (V, E)$ into two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ as in Theorem 6.14. He will play two separate games in parallel; playing on E_1 he will take care of the expansion of large sets, whereas playing on E_2 he will take care of the expansion of small sets.

Starting with the former, let $\mathcal{H}_1$ denote the hypergraph whose vertices are the edges of G_1 and whose hyperedges are the induced bipartite subgraphs of G_1 with parts A and $V \setminus A$ where $\lambda^2 n/d^2 \leq |A| \leq n/2$. Note that, by Theorem 6.14, every hyperedge of $\mathcal{H}_1$ is of size at least $\frac{(1-\varepsilon)d-2\lambda}{4}|A|$. Maker's goal is to claim at least a $(1/2 - \varepsilon)$-fraction of every hyperedge of $\mathcal{H}_1$. In order to prove that he can, we will use the following theorem from [17]:

Theorem 6.15 *Let $\mathcal{F}$ be a finite hypergraph and let $0 < x \leq 1$ be a real number. Two players, called Balancer and Unbalancer, play a $(1, 1)$ game on $\mathcal{F}$ with Balancer being the first player. Unbalancer wins if he claims at least $\frac{1+x}{2}|A|$ vertices of some $A \in \mathcal{F}$; otherwise Balancer wins. If*

$$\sum_{A \in \mathcal{F}} [(1+x)^{1+x}(1-x)^{1-x}]^{-|A|/2} < 1,$$

then Balancer has a winning strategy for this game.

Applying Theorem 6.15 with $x = \varepsilon$ we obtain:

$$
\begin{aligned}
&\sum_{B\in\mathcal{H}_1}\left[(1+x)^{1+x}(1-x)^{1-x}\right]^{-|B|/2}\\
\leq\ & \sum_{t=\lambda^2n/d^2}^{n/2}\binom{n}{t}\left[(1+\varepsilon)^{1+\varepsilon}(1-\varepsilon)^{1-\varepsilon}\right]^{-\left(\frac{(1-\varepsilon)d-4\lambda}{8}\right)t}\\
\leq\ & \sum_{t=\lambda^2n/d^2}^{n/2}\left(\frac{en}{t}\left[(1+\varepsilon)^{1+\varepsilon}(1-\varepsilon)^{1-\varepsilon}\right]^{-\frac{(1-\varepsilon)d-4\lambda}{8}}\right)^t\\
\leq\ & \sum_{t=\lambda^2n/d^2}^{n/2}\exp\left\{1+2\log(d/\lambda)-\frac{(1-\varepsilon)d-4\lambda}{8}\left[(1+\varepsilon)\log(1+\varepsilon)+(1-\varepsilon)\log(1-\varepsilon)\right]\right\}^t\\
\leq\ & \sum_{t=\lambda^2n/d^2}^{n/2}\exp\left\{1+2\log(d/\lambda)-\frac{(1-\varepsilon)d-4\lambda}{8}\left[(1+\varepsilon)\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\varepsilon^{n+1}-(1-\varepsilon)\sum_{n=0}^{\infty}\frac{1}{n+1}\varepsilon^{n+1}\right]\right\}^t\\
\leq\ & \sum_{t=\lambda^2n/d^2}^{n/2}\exp\left\{1+2\log(d/\lambda)-\frac{(1-\varepsilon)d-4\lambda}{8}\left(\varepsilon^2+O(\varepsilon^3)\right)\right\}^t\\
=\ & o(1),
\end{aligned}
$$

where the last equality is valid for an appropriate choice of ε.

Next we prove that, playing on G_2, Maker can guarantee expansion of small sets. Maker's goal in the game on G_2 is to build a graph in which the degree of every vertex is at least a $\frac{1}{4}$-fraction of its degree in G_2. The following easy lemma from [47] asserts that there is a strategy for Maker to achieve this goal.

Lemma 6.16 *In a $(1,1)$ Maker-Breaker game played on the edges of a graph $H=(V,E)$, Maker can build a graph M such that $deg_M(u)\geq\left\lfloor\frac{\lfloor deg_H(u)/2\rfloor}{2}\right\rfloor$ for every vertex $u\in V$.*

Let $A\subseteq V$ be any subset of size $1\leq|A|\leq\lambda^2n/d^2$. It follows from property (1) above that $e_G(A)\leq\frac{d|A|^2}{2n}+\lambda|A|$. Moreover, $e_{G_2}(A,V\setminus A)\geq\frac{(1-\varepsilon)d-2\lambda}{4}|A|$ by Theorem 6.14. If Maker plays according to the strategy guaranteed by Lemma 6.16, then he necessarily claims at least

$$
\begin{aligned}
& \frac{e_{G_2}(A)+e_{G_2}(A,V\setminus A)}{4}-e_{G_2}(A)\\
=\ & \frac{e_{G_2}(A,V\setminus A)-3e_{G_2}(A)}{4}\\
\geq\ & \frac{e_{G_2}(A,V\setminus A)-3e_{G}(A)}{4}\\
\geq\ & \frac{(1-\varepsilon)d|A|-2\lambda|A|-6d|A|^2/n-12\lambda|A|}{16}\\
\geq\ & \frac{\left((1-\varepsilon)\,d-14\lambda-6\lambda^2/d\right)|A|}{16}\\
\geq\ & d|A|/16.1
\end{aligned}
$$

edges of G_2 with exactly one endpoint in A.

Let M denote the graph built by Maker by the end of the game and let $M_1 = M \cap E_1$ and $M_2 = M \cap E_2$. We will now prove that $|N_{M_1}(A)| \geq n/81 - |A|$ for every $A \subseteq V$ of size at least $\lambda^2 n/d^2$ and $|N_{M_2}(A)| \geq \frac{d^2}{300\lambda^2}|A|$ for every $A \subseteq V$ of size $1 \leq |A| \leq \lambda^2 n/d^2$.

Assume for the sake of contradiction that there exists a set $A \subseteq V$ of size at least $\lambda^2 n/d^2$ such that $|N_{M_1}(A)| < n/81 - |A|$. by Maker's strategy for the game on E_1 we have $e_{M_1}(A, N_{M_1}(A)) \geq (1/2 - \varepsilon)\frac{(1-\varepsilon)d-2\lambda}{4}|A| > d|A|/8.1$. However,

$$
\begin{aligned}
e_{M_1}(A, N_{M_1}(A)) &\leq e_G(A, N_{M_1}(A)) \\
&\leq \frac{d|A||N_{M_1}(A)|}{n} + \lambda\sqrt{|A||N_{M_1}(A)|} \\
&\leq \frac{d|A|(n/81 - |A|)}{n} + \lambda\sqrt{|A|(n/81 - |A|)} \\
&\leq d|A|/81 + \lambda|A|\sqrt{\frac{n/81 - |A|}{|A|}} \\
&\leq d|A|\left(\frac{1}{81} + \frac{1}{9}\right) \\
&= d|A|/8.1
\end{aligned}
$$

by property (3) above - a contradiction.

Assume now for the sake of contradiction that there exists a set $A \subseteq V$ of size $1 \leq |A| \leq \lambda^2 n/d^2$ such that $|N_{M_2}(A)| < \frac{d^2}{300\lambda^2}|A|$. By Maker's strategy for the game on E_2 we have $e_{M_2}(A, N_{M_2}(A)) \geq d|A|/16.1$. However,

$$
\begin{aligned}
e_{M_2}(A, N_{M_2}(A)) &\leq e_G(A, N_{M_2}(A)) \\
&\leq \frac{d|A|^2 \frac{d^2}{300\lambda^2}}{n} + \lambda|A|\sqrt{\frac{d^2}{300\lambda^2}} \\
&\leq d|A|\left(\frac{d^2|A|}{300\lambda^2 n} + \frac{1}{\sqrt{300}}\right) \\
&< d|A|/16.1
\end{aligned}
$$

by property (3) above - a contradiction.

$\square$

6.5 Planarity, colorability and minor games on sparse graphs

Before proving the results of this section, we will prove the following lemma which will be useful throughout this section.

Lemma 6.17 *Let $G = (V, E)$ be any graph on n vertices with maximum degree $\leq d$ and assume that $q \geq d/2$. Then, playing a $(1, q)$ game on the edges of G, Breaker can force Maker to build a forest.*

The proof of Lemma 6.17 is a direct application of the following result of Bednarska and Pikhurko.

Theorem 6.18 *([22]) Let $M = (E, F)$ be a matroid with rank function ρ. If $q \geq \max_{\emptyset \neq X \subseteq E} \left\lceil \frac{|X|}{\rho(X)} \right\rceil - 1$ then Breaker, as first or second player, can prevent Maker from building an M-circuit.*

Proof: Breaker's strategy is to prevent Maker from building a circuit. Thus we can use Theorem 6.18 with $M(G)$ being the cycle matroid of G. It suffices to prove that our choice of q satisfies the conditions of Theorem 6.18. Let X be any subset of E, let $X_1, \ldots, X_r$ be the connected components of X where $|X_i| = n_i$ for every $1 \leq i \leq r$. Let ρ denote the rank function of $M(G)$ then $\rho(X_i) = n_i - 1$. If $n_i \leq \frac{d}{2}$ then

$$\frac{|X_i|}{\rho(X_i)} \leq \frac{n_i(n_i-1)}{2(n_i-1)} \leq \frac{d}{4} \leq 1 + q.$$

Otherwise $n_i \geq \frac{d}{2} + 1$ and so

$$\frac{|X_i|}{\rho(X_i)} \leq \frac{n_i d/2}{n_i - 1} \leq \frac{d}{2} + \frac{d/2}{n_i - 1} \leq 1 + q.$$

Thus

$$\max_{\emptyset \neq X \subseteq E} \left\lceil \frac{|X|}{\rho(X)} \right\rceil - 1 = \max_{\emptyset \neq X \subseteq E} \left\lceil \frac{\sum_{1 \leq i \leq r} |X_i|}{\sum_{1 \leq i \leq r} \rho(X_i)} \right\rceil - 1 \leq q$$

and so the lemma follows. □

Proof of Theorem 6.4:

1. Using lemma 6.17 we conclude that Breaker can force Maker to build a forest which is clearly planar.

2. (a) Let G be a d-regular graph on n vertices with girth $\Omega(\log n)$ (such a graph exists for every sufficiently large even n). No matter how he plays, by the end of the game Maker will have at least $\frac{dn}{2(q+1)} \geq (1 + \varepsilon)n$ edges and so his graph will be non-planar.

(b) The proof is a straightforward generalization of the proof of Theorem 2.1 from Chapter 2 (see also [45]).

Let $0 < \varepsilon < 1/3$ (the restriction $\varepsilon < 1/3$ is technical) and let $q \leq \left(\frac{1}{2} - \varepsilon\right) d$, where $d = d(\varepsilon)$ is sufficiently large. Let G be any d-regular graph on n vertices; we will provide a strategy for Maker to build a non-planar subgraph of G. Let $\alpha = \frac{2\varepsilon}{1-2\varepsilon}$ and let $\alpha_d = \alpha_d(\varepsilon)$ be the real number satisfying the equation

$$(1 + \alpha_d)n = \frac{dn/2}{(\frac{1}{2} - \varepsilon)d + 1}.$$

Then $\lim_{d\to\infty} \alpha_d = \alpha$. Let m_n denote the number of edges that Maker will claim by the end of the game on G. We have $m_n - (1 + \frac{\alpha}{2})n = \Omega(n)$.

Let $k = k(\varepsilon)$ be the smallest positive integer such that

$$1 + \frac{\alpha}{2} > \frac{k}{k-2}.$$

Maker's goal is to avoid cycles of length smaller than k, which we will call "short cycles", during the first $(1 + \frac{\alpha}{2})n$ moves. If he succeeds, Maker's graph will at that point of the game have

$$(1 + \frac{\alpha}{2})n > \frac{k}{k-2}n$$

edges and girth at least k. But, it is well-known that a planar graph with girth at least k cannot have more than $\frac{k}{k-2}(n-2)$ edges. Hence, Maker's graph will already be non-planar, and he will win no matter how the game continues.

It remains to show that Maker can indeed avoid claiming a short cycle during the first $(1 + \frac{\alpha}{2})n$ moves. His strategy is the following. For as long as possible he claims edges (u, v) that satisfy the following two properties:

(a) (u, v) does not close a short cycle;

(b) the degrees of both u and v in Maker's graph are less than $d^{1/(k+1)}$.

It suffices to prove that when this is no longer possible, that is, every remaining unclaimed edge violates either (a) or (b), Maker has already claimed at least $(1 + \frac{\alpha}{2})n$ edges.

Every edge that violates property (b) must have at least one endpoint of degree $d^{1/(k+1)}$ in Maker's graph. Since Maker's graph at any point of the game contains at most $(1 + \alpha)n$ edges, there are at most $2(1 + \alpha)nd^{-1/(k+1)}$ vertices of degree at least $d^{1/(k+1)}$. Moreover, by our assumption on G, every vertex is incident with at most d edges. Therefore, the number of edges that violate property (b) is at most

$$d \cdot 2(1 + \alpha)nd^{-1/(k+1)} = o(dn).$$

For any fixed $0 < s < k$ and every vertex v, the number of paths of length s that have v as one endpoint is at most Δ^s, where Δ is the maximum degree in Maker's graph. If we assume that property (b) has not been violated, then $\Delta \leq d^{1/(k+1)}$. Therefore, there are at most

$$n \cdot \sum_{s=3}^{k-1} d^{s/(k+1)} = o(dn)$$

edges that close a short cycle.

Thus, the total number of edges that violate (a) or (b) if claimed by Maker, is $o(dn)$. On the other hand, after $(1 + \frac{\alpha}{2})n$ moves have been played, the number of unclaimed edges is $\Theta(dn)$. Hence, in the first $(1 + \frac{\alpha}{2})n$ moves Maker can claim edges that satisfy properties (a) and (b), which means that he does not claim a short cycle.

(c) Let G be a graph on n vertices with average degree $\geq d$. Regardless of his strategy, at the end of the game Maker will have at least $dn/2(q+1) > 3n - 6$ edges and so his graph will not be planar.

□

The Coloring game

Proof of Theorem 6.5:

1. For sufficiently large d, if we play the $(1, q)$ game on the edges of K_{d+1}, then by Theorem 2.3 from Chapter 2 (see also [45]), Maker has a winning strategy if $q \leq \frac{d+1}{3k \log k}$.

2. Let $G = (V, E)$ be a d-regular graph on n vertices and let r be the smallest integer such that $d \leq r\lfloor k/2 \rfloor$. By Petersen's theorem (see e.g. [31]) there exist pairwise edge disjoint graphs $G_1, G_2, \ldots, G_{\lfloor k/2 \rfloor}$ such that $G = \bigcup_{i=1}^{\lfloor k/2 \rfloor} G_i$ and the maximum degree in every G_i is at most r. Applying Lemma 6.17 we conclude that Breaker can force Maker to build a graph M such that $M \cap G_i$ is a forest for every $1 \leq i \leq \lfloor k/2 \rfloor$. Clearly M is $2\lfloor k/2 \rfloor - 1$ degenerate and is therefore k-colorable.

□

Proof of Theorem 6.6:

1. Let H be any subgraph of G. It is well known (and easy) that $\chi(H)\chi(G \setminus H) \geq r$. Thus, if Maker is the first player then by the *Strategy Stealing* argument, he can build a graph M satisfying $\chi(M) \geq \sqrt{r}$. If Breaker starts the game by claiming the edge e, then we use the *Strategy Stealing* argument on $G \setminus e$. The proposition now follows as clearly $\chi(G \setminus e) \geq r - 1$.

2. Let G be a complete r-partite graph with parts $A_1, \ldots, A_r$, where $|A_i| = m := \binom{r}{2}(1+q)^{\binom{r}{2}-1}$ for every $1 \leq i \leq r$. Let $\mathcal{H}$ be the hypergraph whose vertices are the edges of G and whose hyperedges are the r-cliques of G. We will use the following "weak win" criterion for Maker's win due to Beck.

Theorem 6.19 *[6, Theorem 1.2] Let $\mathcal{F}$ be an n-uniform hypergraph with vertex set V. Let $\Delta_2(\mathcal{F})$ denote the maximum number of hyperedges of $\mathcal{F}$ containing two given vertices. If*

$$|\mathcal{F}| > q^2(1+q)^{n-3}|V|\Delta_2(\mathcal{F})$$

then Maker, as first player, wins the $(1, q)$ game on $\mathcal{F}$.

Clearly $\mathcal{H}$ is $\binom{r}{2}$-uniform, $|\mathcal{H}| = m^r$, $|V(\mathcal{H})| = \binom{r}{2}m^2$ and $\Delta_2(\mathcal{H}) = m^{r-3}$. Hence

$$|\mathcal{H}| = m^r = \binom{r}{2}(1+q)^{\binom{r}{2}-1}m^{r-1} > q^2(1+q)^{\binom{r}{2}-3}|V(\mathcal{H})|\Delta_2(\mathcal{H}).$$

Applying Theorem 6.19, we conclude that Maker will claim an r-clique and so his graph will be r-chromatic.

□

The fixed minor game

Proof of Theorem 6.7:

1. We will use the following lemma (see Lemma 3.14 from Chapter 3 and also [45]).

Lemma 6.20 *Let $G = (V, E)$ be a graph on n vertices, with average degree $2 + \alpha$ for some $\alpha > 0$ and girth $g^* \geq \left(1 + \frac{2}{\alpha}\right)(4 \log_2 t + 2 \log_2 \log_2 t + c)$ where c is an appropriate constant. Then G admits a K_t-minor.*

Let $\varepsilon > 0$ and let G be a graph on n vertices, with girth $\Omega(\log n)$ and average degree $d \geq (2 + \varepsilon)(q + 1)$. Regardless of his strategy, Maker will build a graph that satisfies the conditions of Lemma 6.20 and will thus win.

2. Using lemma 6.17 we conclude that Breaker can force Maker to build a forest, which does not admit a K_t minor for any $t \geq 3$.

□

Proof of Theorem 6.8:

Let t be the positive integer satisfying $t^2 + t = 2n$ (we only consider values of n for which this t is an integer). Let $G = (V, E)$ be the graph obtained from $K_t = (V', E')$ by adding a new vertex in the middle of every edge of K_t, that is $V = V' \cup \{x_e : e \in E'\}$ and $E = \{(u, x_e) : u \in e \in E'\}$. Breaker plays according to the following natural pairing strategy: whenever Maker claims an edge $(u, x_e) \in E$, Breaker claims the edge (v, x_e) where $e = (u, v)$. Clearly, Maker's graph is a forest, which is K_3-minor free. □

6.6 Concluding remarks and open problems

Monotonicity: It was proved in Section 6.2 that $n/\log n \leq C_n(k) \leq n/\log n$ for every $2 \leq k \leq (\log 2 - \varepsilon)n/\log n$. Though, it seems unlikely, it is possible that $C_n(k)$ oscillates between these two bounds. We make the following conjecture:

Conjecture 6.21 *$C_n(k)$ is a monotone function of k, that is $C_n(k) \leq C_n(k+1)$ for every positive integer k.*

Planarity game: It was proved in Theorem 6.4 that if $q < d/2 - 1$, then there exists a d-regular graph on n vertices (for sufficiently large even n) on which Maker wins the $(1, q)$ planarity game; whereas, if $q \geq d/2$, then Breaker wins the $(1, q)$ planarity game on any graph with maximum degree at most d. The case $q = d/2 - 1$ is open. Note that Breaker can win this game on almost every d-regular graph (for even d). Indeed, a.s. a random d-regular graph admits $d/2 = q + 1$ pairwise edge disjoint spanning trees. By Theorem 3.8 from Chapter 3 (see also [42]), Breaker can assume the role of Enforcer and force Maker to build a spanning tree. In the end, Maker's graph will consist of a spanning tree and one additional edge; clearly such a graph is planar.

The general case however, cannot be resolved in this fashion. Indeed, let G be a connected 4-regular graph with "many" bridges. Maker can easily avoid claiming half of them, thus creating many cycles.

Relations between different game models: As was partly indicated in the Introduction, the following implications hold:

$$Random \Rightarrow Sparse \Rightarrow Fast.$$

Indeed, if Maker a.s. wins the $(1, 1, X_p, \mathcal{H}_p)$ game, then in particular there is a positive probability that he will win this game, entailing the existence of a hypergraph $(X^p, \mathcal{H}^p)$ such that $|X^p| \leq |X|p$ and Maker wins the $(1, 1, X^p, \mathcal{H}^p)$ game. Furthermore, if Maker wins the $(1, 1, X, \mathcal{H})$ game, and $(\bar{X}, \bar{\mathcal{H}})$ is any hypergraph that contains $(X, \mathcal{H})$, then Maker can win the $(1, 1, \bar{X}, \bar{\mathcal{H}})$ game within at most $|X|/2$ moves.

Moreover, it was proved in [46] that *Biased* $\Rightarrow$ *Fast*; that is, if Maker wins the $(1, q, X, \mathcal{H})$ game, then he can win the $(1, 1, X, \mathcal{H})$ game within at most $\frac{|X|}{q+1}$ moves.

In [70], it was proved that in some, but not all, cases *Biased* $\Leftrightarrow$ *Random*; that is, Maker wins the $(1, q, X, \mathcal{H})$ game iff he a.s. wins the $(1, 1, X_p, \mathcal{H}_p)$ game for $p \leq c/q$, where c is an appropriate constant. We pose the following question:

Question 6.22 *Is it true that Biased* $\Rightarrow$ *Random? What about Biased* $\Rightarrow$ *Sparse?*

Part III

A new model of Positional Games

Chapter 7

Bart-Moe games

7.1 Introduction

An unbiased positional game is a pair $(X, \mathcal{H})$, where the set X is called the "board", and $\mathcal{H} \subseteq 2^X$ is the family of "winning subsets". During the game two players alternately occupy elements of the board. The first player, called Occupier, wins the game if at the end of the game the subset of the board he occupies is a winning subset, otherwise the second player, called Preventer, wins.

Classical examples of this setting are Maker/Breaker-type games, in which case $\mathcal{H}$ is a monotone increasing family. Maker plays the role of Occupier and Breaker the role of Preventer. Once Maker occupies a minimal element of $\mathcal{H}$ with respect to inclusion, the game can be stopped as Maker has already ensured his win. In fact, sometimes we will include an element of 2^X in $\mathcal{H}$ iff it is a minimal winning subset. Not as well studied but equally interesting is the case of a monotone decreasing $\mathcal{H}$ which corresponds to Avoider/Enforcer-type games. In this case Occupier wins if he *avoids* occupying a member of $2^X \setminus \mathcal{H}$, hence plays Avoider in an Avoider/Enforcer-type game $(X, 2^X \setminus \mathcal{H})$.

Frieze et al. [38] studied positional games where the family of winning sets is the intersection of a monotone increasing family and a monotone decreasing family. Here we generalize their results to biased games, that is, when Occupier occupies p elements of the board per move instead of 1. One of the major motivating ideas behind this approach is to try and create pseudo-random graphs of the appropriate edge-density. These graphs can then be used to prove that numerous other natural games of the Maker/Breaker-type can be won by Maker. We will not discuss here the notion of pseudo-random graphs in much detail. the interested reader is referred to a recent survey [57] on the subject. Very generally speaking, a pseudo-random graph is a graph whose edge distribution resembles closely that of a truly random graph of the same density on the same number of vertices.

Our setting is the following. Let $\mathcal{A}$ and $\mathcal{B}$ be hypergraphs with a common vertex set V. In a $(p, q, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game (consult the Simpsons series for the origin of the names; a more mathematical explanation is given later) the players take turns selecting previously unclaimed vertices of V. The first player, called Bart (to denote his role as Breaker and

Avoider together), selects p vertices per move and the second player, called Moe (to denote his role as Maker or Enforcer), selects q vertices per move. The game ends when every element of V has been claimed by one of the players. Bart wins the game iff he has at least one vertex in every hyperedge $B \in \mathcal{B}$ and no complete hyperedge $A \in \mathcal{A}$. We prove the following sufficient condition for Bart to win the $(p, 1)$-game.

Theorem 7.1 *For hypergraphs $\mathcal{A}$ and $\mathcal{B}$, if*

$$\sum_{A\in\mathcal{A}} \left(1+\frac{1}{p}\right)^{-|A|} + \sum_{B\in\mathcal{B}} (1+p)^{-|B|} < \left(1+\frac{1}{p}\right)^{-p}$$

then Bart has a winning strategy for the $(p, 1, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game.

Remark 1 Theorem 7.1 is a generalization of special cases of several known results. If $\mathcal{A} = \emptyset$ then we get a Maker-Breaker game on $\mathcal{B}$ for which Breaker has a winning strategy if

$$\sum_{B\in\mathcal{B}} (1+p)^{-|B|} < \left(1+\frac{1}{p}\right)^{-p}.$$

This is almost as good (and can be made as good by trivial changes to the proof) as a result of Beck for $q = 1$ (c.f. [6]). If $\mathcal{B} = \emptyset$, then we get an Avoider-Enforcer game on $\mathcal{A}$ for which Avoider has a winning strategy if

$$\sum_{A\in\mathcal{A}} \left(1+\frac{1}{p}\right)^{-|A|} < \left(1+\frac{1}{p}\right)^{-p}.$$

This is the same as Theorem 3.1 from Chapter 3 for the case $q = 1$ (see also [42]). If $\mathcal{A} = \mathcal{B}$ then we get a sufficient condition for the first player to win the $(p, 1, \mathcal{A})$ 2-coloring game. This generalizes a lemma from [38] which applies only to the case $p = 1$.

One of our main motivations to study Bart-Moe games are Maker/Breaker-type positional games played on the edges of the complete graph K_n. In these games, the goal of Maker is usually to build a graph which satisfies some graph theoretic property. Consider for example, following [62], the Maker-Breaker game where Maker's goal is to occupy $\frac{p}{2(p+q)}(1 + o(1))n$ edge-disjoint Hamiltonian cycles. To handle such tasks, often an indirect approach is more fruitful. In our example, instead of concentrating on building the cycles, Maker creates a pseudo-random graph with the appropriate parameters and then shows (or cites the vast literature on pseudo-random graphs) that any such graph contains the required number of edge-disjoint Hamiltonian cycles.

JumbleG We need a few definitions related to pseudo-random graphs. Let $G = (V, E)$, $|V| = n$, be a graph and let $S, T \subseteq V$ be non-empty and disjoint. We say that the pair (S, T) is (α, ε)-*unbiased* if

$$\left|\frac{e_G(S,T)}{|S||T|} - \alpha\right| \le \varepsilon,$$

where $e_G(S,T)$ is the number of edges with one end in S and the other in T. The graph G is said to be *(α, ε)-regular* if its minimum degree is at least $(\alpha - \varepsilon)n$ and any pair S, T of disjoint subsets of V, such that $|S|, |T| \geq \varepsilon n$, is (α, ε)-unbiased (note that this definition is slightly different than the definition given in [38], but they are essentially the same).
In the (p,q) game of JumbleG (c.f. [38]), two players alternately select unclaimed edges of K_n. The first player, called Jumbler (referring to the pseudo-random "jumbled graphs" of Thomason [74]), wins this game iff he is able to build a graph which is $(\frac{p}{p+q}, \varepsilon)$-regular. A similar game, also presented in [38], is (p,q)-JumbleG2, also played on K_n. The first player, called Jumbler, wins this game iff he is able to build a graph with minimum degree at least $(\frac{p}{p+q} - \varepsilon)n$ and maximum co-degree at most $((\frac{p}{p+q})^2 + \varepsilon)n$. The fact that these properties indeed entail pseudo-randomness is discussed in [38]. Using Theorem 7.1 we prove the following generalizations of Theorems 1 and 2 from [38].

Theorem 7.2 *If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$, $\varepsilon \geq 3\sqrt[3]{\frac{\log n}{np}}$ and n is sufficiently large then Jumbler has a winning strategy for the $(p,1)$-JumbleG game.*

Theorem 7.3 *If $p < \frac{1}{16}\sqrt[3]{\frac{n}{\log n}}$, $\varepsilon \geq 8\sqrt{\frac{\log n}{np}}$ and n is sufficiently large then Jumbler has a winning strategy for the $(p,1)$-JumbleG2 game.*

The lower bound on ε given in Theorem 7.2 is tight up to a multiplicative constant factor. In fact, for smaller values of ε, the second player wins $(p,1)$-JumbleG no matter how he plays:

Theorem 7.4 *Let n be a sufficiently large positive integer. For every positive integer $p = o\left(\sqrt{\frac{n}{\log n}}\right)$ and for every $\varepsilon \leq c\sqrt[3]{\frac{\log n}{np}}$, where $c < 1/5$, no graph on n vertices is $(\frac{p}{p+1}, \varepsilon)$-regular.*

Discrepancy In a $(p,q,\mathcal{H})$ ε-Discrepancy game the players alternately select previously unclaimed vertices of a hypergraph $\mathcal{H}$ until every vertex has been claimed by some player. The first player, called Balancer, selects p vertices per move and the second player, called Unbalancer, selects q vertices per move. Let B denote the set of vertices selected by Balancer at the end of the game. If $||B \cap A| - \frac{p}{p+q}|A|| < \varepsilon|A|$ for every $A \in \mathcal{H}$ then Balancer wins the game; otherwise Unbalancer wins. The (1,1) version of the Discrepancy game has been recently considered in [2].

We prove a sufficient condition for Balancer to win this game on uniform hypergraphs for $q = 1$:

Theorem 7.5 *Let $\mathcal{H}$ be an n-uniform hypergraph. If $p < \frac{1}{3}\sqrt[3]{\frac{n}{\log(|\mathcal{H}|n)}}$, $\varepsilon \geq 3\sqrt{\frac{\log(|\mathcal{H}|n)}{np}}$ and n is sufficiently large then Balancer has a winning strategy for the $(p,1,\mathcal{H})$ ε-Discrepancy game.*

For the sake of simplicity and clarity of presentation, we make no effort to optimize the constants in Theorems 7.2, 7.3 and 7.5. We also omit floor and ceiling signs whenever these are not crucial. Throughout this chapter log stands for the natural logarithm.

The rest of this chapter is organized as follows: in Section 7.2 we prove Theorem 7.1. In Section 7.3 we prove Theorems 7.2,7.3 and 7.4, and discuss their applications to several positional games. In Section 7.4 we prove Theorem 7.5. Finally, in Section 7.5 we present some open problems.

7.2 The criterion

Our proof is based on Beck's proof of a sufficient condition for Breaker to win the $(p, q, \mathcal{H})$ Maker-Breaker game [6], which in turn is based on a method of Erdős and Selfridge [36].

Given a hypergraph $\mathcal{A}$ and disjoint subsets X and Y of the vertex set V let $\varphi_1(X, Y, \mathcal{A}) = \sum'_A (1+\frac{1}{p})^{-|A \setminus X|}$ where the summation $\sum'$ is extended over those $A \in \mathcal{A}$ for which $A \cap Y = \emptyset$. Given $z \in V$, let $\varphi_1(X, Y, \mathcal{A}, z) = \sum''_A (1+\frac{1}{p})^{-|A \setminus X|}$ where the summation $\sum''$ is extended over those $A \in \mathcal{A}$ for which $z \in A$ and $A \cap Y = \emptyset$. Similarly, let $\varphi_2(X, Y, \mathcal{B}) = \sum'_B (1+p)^{-|B \setminus Y|}$ where the summation $\sum'$ is extended over those $B \in \mathcal{B}$ for which $B \cap X = \emptyset$. Given $z \in V$ let $\varphi_2(X, Y, \mathcal{B}, z) = \sum''_B (1+p)^{-|B \setminus Y|}$ where the summation $\sum''$ is extended over those $B \in \mathcal{B}$ for which $z \in B$ and $B \cap X = \emptyset$.
Now consider a play according to the rules. Let $x_i^{(1)}, \ldots, x_i^{(p)}$ and y_i denote the vertices chosen by Bart and Moe on their i^{th} move, respectively.
Let $X_i = \{x_1^{(1)}, \ldots, x_1^{(p)}, \ldots, x_i^{(1)}, \ldots, x_i^{(p)}\}$, $Y_i = \{y_1, \ldots, y_i\}$ where $X_0 = Y_0 = \emptyset$.
Furthermore let $X_{i,j} = X_i \cup \{x_{i+1}^{(1)}, \ldots, x_{i+1}^{(j)}\}$ where $X_{i,0} = X_i$.
For every non-negative integer i let $\psi(i) = \psi_1(i) + \psi_2(i)$ where $\psi_1(i) = \varphi_1(X_i, Y_i, \mathcal{A})$ and $\psi_2(i) = \varphi_2(X_i, Y_i, \mathcal{B})$. Bart loses if and only if there exists an integer i such that $A \subseteq X_i$ for some $A \in \mathcal{A}$ or $B \subseteq Y_i$ for some $B \in \mathcal{B}$. In either case $\psi(i) \geq 1$. It follows that if $\psi(i) < 1$ for every $i \geq 0$ then Bart wins the game. Now Bart's strategy is the following: on his i^{th} move, for every $1 \leq k \leq p$, he computes the value of $p\varphi_2(X_{i-1,k-1}, Y_{i-1}, \mathcal{B}, x) - \varphi_1(X_{i-1,k-1}, Y_{i-1}, \mathcal{A}, x)$ for every vertex $x \in V \setminus (Y_{i-1} \cup X_{i-1,k-1})$ and then selects $x_i^{(k)}$ for which the maximum is attained. First, we will prove that $\psi(i+1) \leq \psi(i)$ for every $i \geq 0$. Using the maximum property of $x_{i+1}^{(k)}$ and the simple observations $\varphi_1(X, Y, \mathcal{A}, z_2) \leq \varphi_1(X \cup \{z_1\}, Y, \mathcal{A}, z_2)$ and $\varphi_2(X, Y, \mathcal{B}, z_2) \geq \varphi_2(X \cup \{z_1\}, Y, \mathcal{B}, z_2)$, we get

$$\begin{aligned} p\varphi_2(X_{i,k-1}, Y_i, \mathcal{B}, x_{i+1}^{(k)}) - \varphi_1(X_{i,k-1}, Y_i, \mathcal{A}, x_{i+1}^{(k)}) &\geq \\ p\varphi_2(X_{i,k-1}, Y_i, \mathcal{B}, y_{i+1}) - \varphi_1(X_{i,k-1}, Y_i, \mathcal{A}, y_{i+1}) &\geq \\ p\varphi_2(X_{i+1}, Y_i, \mathcal{B}, y_{i+1}) - \varphi_1(X_{i+1}, Y_i, \mathcal{A}, y_{i+1}) & \end{aligned}$$

for every $1 \le k \le p$. So we conclude

$$\begin{aligned}
\psi(i+1) &= \psi_1(i) + \frac{1}{p}\sum_{k=1}^{p} \varphi_1(X_{i,k-1}, Y_i, \mathcal{A}, x_{i+1}^{(k)}) - \varphi_1(X_{i+1}, Y_i, \mathcal{A}, y_{i+1}) \\
&+ \psi_2(i) - \sum_{k=1}^{p} \varphi_2(X_{i,k-1}, Y_i, \mathcal{B}, x_{i+1}^{(k)}) + p\varphi_2(X_{i+1}, Y_i, \mathcal{B}, y_{i+1}) \\
&= \psi(i) + p\varphi_2(X_{i+1}, Y_i, \mathcal{B}, y_{i+1}) - \varphi_1(X_{i+1}, Y_i, \mathcal{A}, y_{i+1}) \\
&- \frac{1}{p}\sum_{k=1}^{p}(p\varphi_2(X_{i,k-1}, Y_i, \mathcal{B}, x_{i+1}^{(k)}) - \varphi_1(X_{i,k-1}, Y_i, \mathcal{A}, x_{i+1}^{(k)})) \le \psi(i).
\end{aligned}$$

By our assumption $\psi(0) < (1+\frac{1}{p})^{-p}$ and so $\psi(i) < 1$ for every integer i except maybe for $i = r$ which denotes the last round of the game. In this round it is possible that only the first player will participate, but then $\psi(r) \le (1+\frac{1}{p})^p \psi_1(r-1) + \psi_2(r-1) \le (1+\frac{1}{p})^p \psi(r-1) \le (1+\frac{1}{p})^p \psi(0) < 1$ and the theorem follows. □

7.3 Winning in JumbleG

The following lemma will be useful in the proofs of Theorems 7.2, 7.3 and 7.5:

Lemma 7.6 *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph such that $|V| = N$ and $|E| = M$. Let $0 < l < \frac{k}{3}$ be an integer. Then there exists a collection $\mathcal{X}$ of l-subsets of V, of size at most $s = \left(\frac{N}{k}\right)^l \log M \min\{\exp\{\frac{l^2}{k}\}, \exp\{\frac{l^2}{2k} + \frac{l^3}{k^2} - \frac{l^2}{2N}\}\}$ such that every hyperedge of $\mathcal{H}$ contains an element of $\mathcal{X}$.*

Proof: Choose s subsets of V, each of size l, randomly, independently and with replacement, and denote the resulting collection by $\mathcal{X}$. By a simple union bound argument we have
$Pr[\exists e \in E \text{ such that } \forall x \in \mathcal{X}, x \nsubseteq e] \le M \left(1 - \frac{\binom{k}{l}}{\binom{N}{l}}\right)^s$.
We will prove that this probability is strictly less than 1.

$$\begin{aligned}
\frac{\binom{k}{l}}{\binom{N}{l}} &= \prod_{i=0}^{l-1} \frac{k-i}{N-i} = \left(\frac{k}{N}\right)^l \prod_{i=0}^{l-1}\left(\frac{1-\frac{i}{k}}{1-\frac{i}{N}}\right) > \left(\frac{k}{N}\right)^l \prod_{i=0}^{l-1}\left(1 - \frac{i}{k}\right) \\
&> \left(\frac{k}{N}\right)^l \exp\{-\sum_{i=0}^{l-1}\frac{2i}{k}\} > \left(\frac{k}{N}\right)^l \exp\{-\frac{l^2}{k}\},
\end{aligned}$$

where the second inequality follows since $1 - x > e^{-2x}$ for every $0 < x < \frac{1}{3}$.

Similarly, and since $e^{-x-x^2} < 1 - x < e^{-x}$ for every $0 < x < \frac{1}{3}$, we have

$$\begin{aligned}
\frac{\binom{k}{l}}{\binom{N}{l}} &= \left(\frac{k}{N}\right)^l \prod_{i=0}^{l-1} \left(\frac{1-\frac{i}{k}}{1-\frac{i}{N}}\right) > \left(\frac{k}{N}\right)^l \exp\{\sum_{i=0}^{l-1} \frac{i}{N} - \sum_{i=0}^{l-1} (\frac{i}{k} + \frac{i^2}{k^2})\} \\
&> \left(\frac{k}{N}\right)^l \exp\{\frac{l^2}{2N} - \frac{l^2}{2k} - \frac{l^3}{k^2}\}.
\end{aligned}$$

Either way, $Pr[\exists e \in E$ such that $\forall x \in \mathcal{X}, x \not\subseteq e] < M \exp\{-s\binom{k}{l}/\binom{N}{l}\} < 1$, and so there exists a collection $\mathcal{X}$ with the desired properties. $\square$

The following technical lemma will save us some calculations later on:

Lemma 7.7 *Let m, r and p be positive integer-valued functions of n, such that $mr \to \infty$. Let ε also be a function of n such that $\varepsilon > 3\sqrt{\frac{\log(mr)}{rp}}$ and $p = o(\varepsilon^{-1})$. Furthermore we define $l = \frac{\varepsilon r}{2}$ and $k = (\frac{1}{p+1} + \varepsilon)r$. Then*

$$m \left(\frac{r}{k}\right)^l \log \binom{r}{k} \exp\{\frac{l^2}{k}\} (1+p)^{-l} = o(1).$$

Proof:

$$\begin{aligned}
m \left(\frac{r}{k}\right)^l \log \binom{r}{k} \exp\left\{\frac{l^2}{k}\right\} (1+p)^{-l} &\leq mr \left(\frac{r}{k}\right)^l \exp\left\{\frac{l^2}{k}\right\} (1+p)^{-l} \\
&\leq mr \left(\left(\frac{1}{p+1} + \varepsilon\right)(1+p)\right)^{-l} \exp\left\{\frac{l^2(p+1)}{r}\right\} \\
&= mr\,(1 + (p+1)\varepsilon)^{-l} \exp\left\{\frac{l^2(p+1)}{r}\right\} \\
&\leq mr \exp\left\{\frac{\varepsilon^2 r^2 (p+1)}{4r} - (1 - o(1))\frac{\varepsilon^2 r(p+1)}{2}\right\} \\
&= o(1),
\end{aligned}$$

where the last equality follows by our choice of ε and the last inequality follows since $p = o(\varepsilon^{-1})$. $\square$

Proof of Theorem 7.2

We will define an auxiliary Bart-Moe game on the edges of K_n, such that Jumbler, playing in the role of Bart, will win JumbleG once he wins the auxiliary game. We will apply Theorem 7.1 to provide the winning strategy.

Let us set $\varepsilon = 3\sqrt[3]{\frac{\log n}{np}}$. For larger ε the statement then trivially follows. Note that by the bound on p, we have $p\varepsilon = o(1)$; this will be used several times in the proof.

Let $G=(V,E)$ where $V=V(K_n)$ and E is the set of all edges claimed by Jumbler. In order to win, Jumbler would like G to be $(\frac{p}{p+1},\varepsilon)$-regular. In particular he would like the pair (S,T) to be $(\frac{p}{p+1},\varepsilon)$-unbiased for every disjoint $S,T\subseteq V$, both of size at least $t=\varepsilon n$. By an averaging argument we can assume that both S and T are of size exactly t. Indeed, let $S',T'\subseteq V$ be disjoint and of size at least t. The expectation of $\frac{e_G(S,T)}{t^2}$, where S and T are random t-subsets of S' and T' respectively is $\frac{e_G(S',T')}{|S'||T'|}$. Clearly if $|\frac{e_G(S,T)}{t^2}-\frac{p}{p+1}|\leq\varepsilon$ for every disjoint pair S,T with $|S|=|T|=\varepsilon n$, then so is the expectation.
Let $\mathcal{T}$ consist of all pairs (S,T) of disjoint subsets of V, both of size exactly t. Fix a pair $(S,T)\in\mathcal{T}$. Jumbler would like to have "many" $S-T$ edges (plays as Breaker), but not "too many" (plays as Avoider). Starting with the latter, let $\mathcal{H}^A_{S,T}=(V^A_{S,T},E^A_{S,T})$ be the hypergraph whose vertices are the edges of K_n with one end in S and the other in T, and whose hyperedges are all the subsets of $V^A_{S,T}$ of size $k_1=(\frac{p}{p+1}+\varepsilon)t^2$. Jumbler would like to avoid claiming a complete $e\in E^A_{S,T}$.
By Lemma 7.6, there exists an s_1-sized collection $\mathcal{X}^A_{S,T}$ of l_1-subsets of $V^A_{S,T}$, where

$$l_1=3n\log n,\ s_1\leq\left(\frac{t^2}{k_1}\right)^{l_1}\log|E^A_{S,T}|\exp\left\{\frac{l_1^2}{2k_1}+\frac{l_1^3}{k_1^2}-\frac{l_1^2}{2|V^A_{S,T}|}\right\},\tag{7.1}$$

such that every $e\in E^A_{S,T}$ contains an element of $\mathcal{X}^A_{S,T}$.

Similarly, let $\mathcal{H}^B_{S,T}=(V^B_{S,T},E^B_{S,T})$ be the hypergraph whose vertices are the edges of K_n with one end in S and the other in T, and whose hyperedges are all the subsets of $V^B_{S,T}$ of size $k_2=(\frac{1}{p+1}+\varepsilon)t^2$. Jumbler would like to claim an element of every $e\in E^B_{S,T}$.
By Lemma 7.6, there exists an s_2-sized collection $\mathcal{X}^B_{S,T}$ of l_2-subsets of $V^B_{S,T}$, where

$$l_2=\frac{3n\log n}{p},\ s_2\leq\left(\frac{t^2}{k_2}\right)^{l_2}\log|E^B_{S,T}|\exp\left\{\frac{l_2^2}{k_2}\right\},\tag{7.2}$$

such that every $e\in E^B_{S,T}$ contains an element of $\mathcal{X}^B_{S,T}$.

Jumbler would also like to have $deg_G(u)\geq(\frac{p}{p+1}-\varepsilon)n$ for every $u\in V$. For a vertex $u\in V$ let $\mathcal{H}_u=(V_u,E_u)$ be the hypergraph whose vertices are the edges of K_n incident with u, and whose hyperedges are all the subsets of V_u of size $k_3=(\frac{1}{p+1}+\varepsilon)n$. Jumbler would like to claim an element of every $e\in E_u$.
By Lemma 7.6, there exists an s_3-sized collection $\mathcal{X}_u$ of l_3-subsets of V_u, where

$$l_3=\frac{\varepsilon n}{2},\ s_3\leq\left(\frac{n}{k_3}\right)^{l_3}\log|E_u|\exp\left\{\frac{l_3^2}{k_3}\right\},\tag{7.3}$$

such that every $e\in E_u$ contains an element of $\mathcal{X}_u$.

Now we are ready to define our auxiliary game. Let $\mathcal{A}=\bigcup_{(S,T)\in\mathcal{T}}\mathcal{X}^A_{S,T}$, $\mathcal{B}_1=\bigcup_{(S,T)\in\mathcal{T}}\mathcal{X}^B_{S,T}$ and $\mathcal{B}_2=\bigcup_{u\in V}\mathcal{X}_u$. If Bart can win the $(p,1,\mathcal{A}\cup(\mathcal{B}_1\cup\mathcal{B}_2))$ Bart-Moe game, then Jumbler has a winning strategy for the $(p,1)$ JumbleG game on K_n. By Theorem 7.1 it suffices to prove that

$$\sum_{A\in\mathcal{A}}\left(1+\frac{1}{p}\right)^{-|A|}+\sum_{B\in\mathcal{B}_1}(1+p)^{-|B|}+\sum_{B\in\mathcal{B}_2}(1+p)^{-|B|}<\frac{1}{e}.$$

By (7.3) and Lemma 7.7 (with $m=n$, $r=n$, $k=k_3$ and $l=l_3$), $\sum_{B\in\mathcal{B}_2}(1+p)^{-|B|}=o(1)$ and so it suffices to prove that $\sum_{A\in\mathcal{A}}\left(1+\frac{1}{p}\right)^{-|A|}=o(1)$ and $\sum_{B\in\mathcal{B}_1}(1+p)^{-|B|}=o(1)$. By (7.1) and since $p=o(\varepsilon^{-1})$ we have:

$$\begin{aligned}
&\sum_{A\in\mathcal{A}}\left(1+\frac{1}{p}\right)^{-|A|}\leq\binom{n}{t}^2 s_1\left(1+\frac{1}{p}\right)^{-l_1}\\
<\;& n^{2t}\left(\frac{t^2}{k_1}\right)^{l_1}\log\left(\frac{t^2}{k_1}\right)\exp\left\{\frac{l_1^2}{2k_1}+\frac{l_1^3}{k_1^2}-\frac{l_1^2}{2t^2}\right\}\left(1+\frac{1}{p}\right)^{-l_1}\\
<\;& n^{2t}\left(\left(\frac{p}{p+1}+\varepsilon\right)\left(1+\frac{1}{p}\right)\right)^{-l_1}t^2\exp\left\{\frac{l_1^3}{k_1^2}+\frac{l_1^2}{2t^2}\left(\frac{1}{\frac{p}{p+1}+\varepsilon}-1\right)\right\}\\
\leq\;& t^2n^{2t}(1+\varepsilon)^{-l_1}\exp\left\{2\frac{l_1^2}{2t^2p}\right\}\\
\leq\;& t^2n^{2t}\exp\left\{\frac{l_1^2}{t^2p}\right\}\exp\{-(1-o(1))\varepsilon l_1\}\\
\leq\;& n^{2\varepsilon n+2}n^{\frac{9\log n}{\varepsilon^2p}-3(1-o(1))\varepsilon n}\\
=\;& o(1),
\end{aligned}$$

where the last equality follows from our choice of ε. The fourth inequality follows since $\frac{l_1^3}{k_1^2}<\frac{l_1^2}{2t^2p}$, as can be shown by a straightforward calculation. This is how we get the upper bound on p.
Similarly, by (7.2) and since $p=o(\varepsilon^{-1})$ we have:

$$\begin{aligned}
&\sum_{B\in\mathcal{B}_1}(1+p)^{-|B|}\leq\binom{n}{t}^2 s_2(1+p)^{-l_2}\\
<\;& n^{2t}\left(\frac{t^2}{k_2}\right)^{l_2}\log\left(\frac{t^2}{k_2}\right)\exp\left\{\frac{l_2^2}{k_2}\right\}(1+p)^{-l_2}\\
<\;& n^{2t}\left(\left(\frac{1}{p+1}+\varepsilon\right)(1+p)\right)^{-l_2}t^2\exp\left\{\frac{l_2^2(p+1)}{t^2}\right\}\\
=\;& t^2n^{2t}(1+(p+1)\varepsilon)^{-l_2}\exp\left\{\frac{l_2^2(p+1)}{t^2}\right\}\\
\leq\;& t^2n^{2t}\exp\left\{\frac{l_2^2(p+1)}{t^2}\right\}\exp\{-(1-o(1))\varepsilon(p+1)l_2\}\\
\leq\;& n^{2\varepsilon n+2}n^{\frac{9\log n}{\varepsilon^2p}-3(1-o(1))\varepsilon n}\\
=\;& o(1),
\end{aligned}$$

where the last equality follows from our choice of ε. $\square$

Proof of Theorem 7.3

Again, we will define an auxiliary Bart-Moe game such that Bart's win in this auxiliary game implies Jumbler's win in JumbleG2.

We set $\varepsilon = 8\sqrt{\frac{\log n}{np}}$. Then $p\varepsilon = o(1)$ by the upper bound on p.

Let $G = (V, E)$ where $V = V(K_n)$ and E is the set of all edges claimed by Jumbler. In order to win, Jumbler would like G to have minimum degree at least $(\frac{p}{p+1} - \varepsilon)n$ (plays as Breaker) and maximum co-degree at most $((\frac{p}{p+1})^2 + \varepsilon)n$ (plays as Avoider). Starting with the latter, for every two vertices $u, w \in V$ and every set $S \subseteq V \setminus \{u, w\}$ of size $k_1 = ((\frac{p}{p+1})^2 + \varepsilon)n$, Jumbler would like to avoid claiming the set of edges $\{(x, y) | x \in S, y \in \{u, w\}\}$.

For every two vertices $u, w \in V$ define a hypergraph $\mathcal{H}_{u,w} = (V_{u,w}, E_{u,w})$ where $V_{u,w} = V \setminus \{u, w\}$ and $E_{u,w}$ is the set of all subsets of $V_{u,w}$ of size k_1. By Lemma 7.6, there exists an s_1-sized collection $\mathcal{X}_{u,w}$ of l_1-subsets of $V_{u,w}$, where

$$l_1 = \frac{\varepsilon np}{2},\ s_1 \leq \left(\frac{n}{k_1}\right)^{l_1} \log |E_{u,w}| \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\}, \tag{7.4}$$

such that every $e \in E_{u,w}$ contains an element of $\mathcal{X}_{u,w}$.

Jumbler would also like to have $deg_G(u) \geq (\frac{p}{p+1} - \varepsilon)n$ for every $u \in V$. Fix $u \in V$ and let $\mathcal{H}_u = (V_u, E_u)$ be the hypergraph whose vertices are the edges of K_n incident with u, and whose hyperedges are all the subsets of V_u of size $k_2 = (\frac{1}{p+1} + \varepsilon)n$. Jumbler would like to claim an element of every $e \in E_u$.

By Lemma 7.6, there exists an s_2-sized collection $\mathcal{X}_u$ of l_2-subsets of V_u, where

$$l_2 = \frac{\varepsilon n}{2},\ s_2 \leq \left(\frac{n}{k_2}\right)^{l_2} \log |E_u| \exp\left\{\frac{l_2^2}{k_2}\right\}, \tag{7.5}$$

such that every $e \in E_u$ contains an element of $\mathcal{X}_u$.

We can now define our auxiliary Bart-Moe game. Let $\mathcal{B} = \bigcup_{u \in V} \mathcal{X}_u$ and

$$\mathcal{A} = \bigcup_{\substack{u,w \in V \\ u \neq w}} \{\{(u, x) | x \in e\} \cup \{(w, x) | x \in e\} : e \in \mathcal{X}_{u,w}\}.$$

If Jumbler can win the $(p, 1, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game as Bart, then he has a winning strategy for the $(p, 1)$ JumbleG2 game on K_n. By Theorem 7.1 it suffices to prove that

$$\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} + \sum_{B \in \mathcal{B}} (1 + p)^{-|B|} < \frac{1}{e}.$$

By (7.5) and Lemma 7.7 (with $m = n$, $r = n$,$k = k_2$ and $l = l_2$), $\sum_{B\in\mathcal{B}} (1+p)^{-|B|} = o(1)$ and so it suffices to prove that
$\sum_{A\in\mathcal{A}} \left(1+\frac{1}{p}\right)^{-|A|} = o(1)$. By (7.4) and since $p = o(\varepsilon^{-1})$ we have:

$$\begin{aligned}
&\sum_{A\in\mathcal{A}} \left(1+\frac{1}{p}\right)^{-|A|} \leq \binom{n}{2} s_1 \left(1+\frac{1}{p}\right)^{-2l_1} \\
\leq\ & \frac{n^2}{2}\left(\frac{n}{k_1}\right)^{l_1} \log\binom{n}{k_1} \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\}\left(1+\frac{1}{p}\right)^{-2l_1} \\
\leq\ & \frac{n^3}{2}\left(\left(\left(\frac{p}{p+1}\right)^2 + \varepsilon\right)\left(1+\frac{1}{p}\right)^2\right)^{-l_1} \exp\left\{\frac{l_1^3}{k_1^2} + \frac{l_1^2}{2n}\left(\frac{1}{\left(\frac{p}{p+1}\right)^2 + \varepsilon} - 1\right)\right\} \\
\leq\ & \frac{n^3}{2}(1+\varepsilon)^{-l_1} \exp\left\{\frac{3l_1^2}{2np} + \frac{l_1^2}{4np}\right\} \\
\leq\ & \frac{n^3}{2} \exp\left\{\frac{7\varepsilon^2 n^2 p^2}{16np} - (1-o(1))\frac{\varepsilon^2 np}{2}\right\} \\
=\ & o(1),
\end{aligned}$$

where the last equality follows from our choice of ε. The fourth inequality follows since $\frac{l_1^3}{k_1^2} < \frac{l_1^2}{4np}$, as can be shown by a straightforward calculation. This is how we get the upper bound on p. $\square$

7.3.1 The tightness of Theorem 7.2

Proof of Theorem 7.4
Let $G = (V,E)$ be any graph on n vertices. It suffices to prove that there exist disjoint sets $S, T \subseteq V$, both of size $t = \varepsilon n$, such that the pair (S,T) is not $(\frac{1}{p+1}, \varepsilon)$-unbiased (indeed such a pair (S,T) is (α,ε)-unbiased in a graph iff it is $(1-\alpha,\varepsilon)$-unbiased in the complement of that graph). Assume that $\varepsilon = c\sqrt[3]{\frac{\log n}{np}}$ (this is clearly legitimate as if G is not $(\frac{p}{p+1},\varepsilon)$-regular then it is not $(\frac{p}{p+1},\varepsilon')$-regular for any $\varepsilon' \leq \varepsilon$). Let X be a random t-subset of V chosen uniformly. For every $y \in V$ let $A_{X,y}$ be the event "$y \in V \setminus X$ and $||N(y)\cap X| - \frac{t}{p+1}| > \varepsilon t$", where $N(y) = \{u \in V | (u,y) \in E\}$.

Claim 7.8 *$Pr[A_{X,y}] \geq \frac{2t}{n}$ for every $y \in V$.*

Proof of Claim 7.8 Let $d = d(y)$ denote the degree of y in G. Assume that $d \leq \frac{n-1}{p+1}$. We wish to find a lower bound on

$$Pr[y \in V \setminus X, |N(y)\cap X| \leq \frac{t}{p+1} - \varepsilon t] = \sum_{i=0}^{\frac{t}{p+1}-\varepsilon t} \binom{d}{i}\binom{n-1-d}{t-i}\binom{n}{t}^{-1}. \tag{7.6}$$

A lower bound on $Pr[y \in V \setminus X, |N(y) \cap X| \geq \frac{t}{p+1} + \varepsilon t]$ for $d \geq \frac{n-1}{p+1}$ will follow by an analogous argument. Note that by our choice of p, the sum on the right hand side of (7.6) is not empty. The probability (7.6) is decreasing as a function of d (as for larger values of d it is more likely that y will have many neighbors in X) and so it suffices to bound it for $d = \frac{n-1}{p+1}$. For every $\varepsilon t \leq k \leq \frac{t}{p+1}$ let s_k be the summand corresponding to $i = \frac{t}{p+1} - k$ in (7.6). First, we will estimate

$$s'_k = \binom{\frac{n-1}{p+1}}{\frac{t}{p+1} - k}\binom{n-1-\frac{n-1}{p+1}}{t - \frac{t}{p+1} + k}\binom{n-1}{t}^{-1}.$$

Let $R \sim H\left(t; \frac{n-1}{p+1}, n-1\right)$ be a random variable with a hypergeometric distribution, that is, $R = |A \cap B|$, where A is a fixed $\frac{n-1}{p+1}$-subset of a given set C of size $n-1$, and B is formed by drawing t elements of C at random without replacement. Then $\mu = \mathbb{E}[R] = \frac{t}{p+1}$ and $\sigma^2 = Var(R) \leq \frac{t}{p+1}$. By Chebychev's inequality we have

$$Pr[|R-\mu| \leq 2\sigma] = 1 - Pr[|R-\mu| > 2\sigma] \geq 3/4. \tag{7.7}$$

The function $p(x) = Pr[R = x]$ attains its maximum value at $x = \mu$ (or more accurately at the upper or lower integer part of μ) and so by (7.7) we have $s'_0 = Pr[R = \frac{t}{p+1}] \geq \frac{3}{4} \cdot \frac{1}{4\sigma} \geq \frac{1}{6\sigma} \geq \sqrt{\frac{p+1}{36t}}$. For every $0 \leq k \leq 2\varepsilon t$ we have

$$\begin{aligned}
\frac{s'_{k+1}}{s'_k} &= \frac{\binom{\frac{n-1}{p+1}}{\frac{t}{p+1}-k-1}\binom{n-1-\frac{n-1}{p+1}}{t-\frac{t}{p+1}+k+1}}{\binom{\frac{n-1}{p+1}}{\frac{t}{p+1}-k}\binom{n-1-\frac{n-1}{p+1}}{t-\frac{t}{p+1}+k}} \\
&= \frac{(t-k(p+1))\left(n-1-\frac{n-1}{p+1}-t+\frac{t}{p+1}-k\right)}{(n-1-t+(p+1)(k+1))\left(t-\frac{t}{p+1}+k+1\right)} \\
&\geq \frac{t-(p+1)k}{\frac{pt}{p+1}+k+1} \cdot \frac{p}{p+1}\left(1-\frac{3(p+1)(k+1)}{n}\right) \\
&= \frac{t-(p+1)k}{t+\frac{(p+1)(k+1)}{p}}\left(1-\frac{3(p+1)(k+1)}{n}\right) \\
&\geq \left(1-\frac{2(p+1)(k+1)}{t}\right)\left(1-\frac{3(p+1)(k+1)}{n}\right) \\
&\geq 1-\frac{3(p+1)(k+1)}{t},
\end{aligned}$$

where the second equality follows by a straightforward calculation and the first and last inequalities follow since $t = o(n)$.

Now, for every $0 \leq k \leq 2\varepsilon t$ we have

$$s'_k = s'_0 \prod_{j=0}^{k-1} \frac{s'_{j+1}}{s'_j} \geq s'_0 \prod_{j=0}^{2\varepsilon t-1} \frac{s'_{j+1}}{s'_j} \geq s'_0\left(1-\frac{6(p+1)\varepsilon t}{t}\right)^{2\varepsilon t} \geq s'_0 \exp\left\{-15(p+1)\varepsilon^2 t\right\}.$$

Moreover $\frac{s_k}{s'_k} = \frac{n-t}{n} = 1 - \varepsilon$ and so

$$\begin{aligned}\sum_{k=\varepsilon t}^{\frac{t}{p+1}} s_k &\geq \sum_{k=\varepsilon t}^{2\varepsilon t} s'_k(1-\varepsilon) \geq (1-\varepsilon)\varepsilon t s'_0 \exp\left\{-15(p+1)\varepsilon^2 t\right\} \\ &\geq (1-\varepsilon)\varepsilon\frac{1}{3}\sqrt{t}\exp\left\{-15(p+1)\varepsilon^3 n\right\} \geq \frac{2t}{n},\end{aligned}$$

where the last inequality follows by our choice of c. This concludes the proof of the claim. □

Let Y_X consist of the vertices $y \in V$ for which $A_{X,y}$ holds. By Claim 7.8 we have $E(|Y_X|) = \sum_{y\in V} E(A_{X,y}) \geq 2t$ and so there exists a t-subset S of V such that $|Y_S| \geq 2t$. Assume without loss of generality that $|N(y) \cap S| < \frac{t}{p+1} - \varepsilon t$ for at least half the vertices of Y_S. Let $T \subseteq Y_S$ consist of any t of these vertices, then the pair (S,T) is not $(\frac{1}{p+1}, \varepsilon)$-unbiased. □

7.3.2 Applications

From Theorems 7.2 and 7.3 we immediately get generalizations of all the corollaries obtained in [38] (the bounds on p result from the use of Theorems 7.2 and 7.3):

- If $p < \frac{1}{16}\sqrt[3]{\frac{n}{\log n}}$ then Maker can build a graph with minimum degree at least $\frac{pn}{p+1} - 8\sqrt{\frac{n\log n}{p}}$.
- If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$ then Maker can build a $\left(\frac{pn}{p+1} - 8\sqrt{\frac{n\log n}{p}}\right)$ vertex connected graph.
- If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$ then Maker can build a graph that contains at least $(\frac{p}{2(p+1)} - 3\varepsilon)n$ edge disjoint hamiltonian cycles for every $\varepsilon > 10(\frac{\log n}{n})^{\frac{1}{6}}$.
- If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$ then Maker can build an r-universal graph, in the sense that it contains an induced copy of every graph on r vertices, for $r = (1+o(1))\log_{p+1} n$. Note that r is in inverse ratio to p as when Maker's graph gets more dense it's harder to find sparse induced subgraphs in it.

We omit the straightforward proofs.

7.4 Biased discrepancy games

Proof of Theorem 7.5:

Let us fix $\varepsilon = 3\sqrt{\frac{\log(|\mathcal{H}|n)}{np}}$. In order to win the game, Balancer would like to have "many" vertices in every hyperedge of $\mathcal{H}$ (plays as Breaker), but not "too many" (plays as Avoider). Starting with the latter, for every $e \in \mathcal{H}$ define a hypergraph $\mathcal{H}_e^A = (V_e^A, E_e^A)$ where V_e^A is the set of vertices of e and E_e^A is the set of all subsets of V_e^A of size $k_1 = (\frac{p}{p+1} + \varepsilon)n$. By Lemma 7.6, there exists an s_1-sized collection $\mathcal{X}_e^A$ of l_1-subsets of V_e^A, where

$$l_1 = \frac{\varepsilon np}{2},\ s_1 \leq \left(\frac{n}{k_1}\right)^{l_1} \log|E_e^A| \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\}, \tag{7.8}$$

such that every hyperedge of $\mathcal{H}_e^A$ contains an element of $\mathcal{X}_e^A$.

Similarly, for every $e \in \mathcal{H}$ define a hypergraph $\mathcal{H}_e^B = (V_e^B, E_e^B)$ where V_e^B is the set of vertices of e and E_e^B is the set of all subsets of V_e^B of size $k_2 = (\frac{1}{p+1} + \varepsilon)n$. By Lemma 7.6, there exists an s_2-sized collection $\mathcal{X}_e^B$ of l_2-subsets of V_e^B, where

$$l_2 = \frac{\varepsilon n}{2},\ s_2 \leq \left(\frac{n}{k_2}\right)^{l_2} \log|E_e^B| \exp\left\{\frac{l_2^2}{k_2}\right\}, \tag{7.9}$$

such that every hyperedge of $\mathcal{H}_e^B$ contains an element of $\mathcal{X}_e^B$.

Let $\mathcal{A} = \bigcup_{e\in\mathcal{H}} \mathcal{X}_e^A$ and $\mathcal{B} = \bigcup_{e\in\mathcal{H}} \mathcal{X}_e^B$. If Balancer, playing as Bart, can win the $(p, 1, \mathcal{A}\cup \mathcal{B})$ Bart-Moe game, then he has a winning strategy for the $(p, 1, \mathcal{H})$ ε-Discrepancy game. By Theorem 7.1 it suffices to prove that

$$\sum_{A\in\mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} + \sum_{B\in\mathcal{B}} (1+p)^{-|B|} < \frac{1}{e}.$$

By (7.9) and Lemma 7.7 (with $m = |\mathcal{H}|$, $r = n$, $k = k_2$ and $l = l_2$), $\sum_{B\in\mathcal{B}} (1+p)^{-|B|} = o(1)$ and so it suffices to prove that
$\sum_{A\in\mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} = o(1)$. By (7.8) and since $p = o(\varepsilon^{-1})$ we have:

$$\begin{aligned}
\sum_{A\in\mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} &\leq |\mathcal{H}| \left(\frac{n}{k_1}\right)^{l_1} \log\binom{n}{k_1} \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\} \left(1 + \frac{1}{p}\right)^{-l_1} \\
&\leq |\mathcal{H}|n \left(\left(\frac{p}{p+1} + \varepsilon\right)\left(1 + \frac{1}{p}\right)\right)^{-l_1} \exp\left\{\frac{l_1^3}{k_1^2} + \frac{l_1^2}{2n}\left(\frac{1}{\frac{p}{p+1} + \varepsilon} - 1\right)\right\} \\
&\leq |\mathcal{H}|n\,(1+\varepsilon)^{-l_1} \exp\{2\frac{l_1^2}{2np}\} \\
&\leq |\mathcal{H}|n \exp\left\{\frac{\varepsilon^2 n^2 p^2}{4np} - (1 - o(1))\frac{\varepsilon^2 np}{2}\right\} \\
&= o(1),
\end{aligned}$$

where the last equality follows from our choice of ε. The third inequality follows since $\frac{l_1^3}{k_1^2} < \frac{l_1^2}{2np}$ by the upper bound on p. $\square$

7.5 Concluding remarks and open problems

- It would be interesting to find a sufficient condition for Bart to win the (p, q) Bart-Moe game for $q > 1$, and apply it to several specific combinatorial games.
- It would be interesting to analyze $(p, 1)$-JumbleG, JumbleG2 and Discrepancy for every value of p. Note that we can consider greater values of p at the cost of enlarging ε; that is, if $m \leq p = o(m^2)$, where m denotes the upper bound on p given in Theorem 7.2, then by a similar argument we can prove that the assertion of Theorem 7.2 holds for $\varepsilon > const \sqrt[3]{\frac{\log n}{n\sqrt{p}}}$. The same can be done with Theorems 7.3 and 7.5. Note that having a certain upper bound on p is reasonable, as $\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|}$ grows with p.

Part IV

From positional games outwards

Chapter 8

A new criterion for Hamiltonicity

8.1 Introduction

A Hamilton cycle in a graph G is a cycle passing through all vertices of G. A graph is called *Hamiltonian* if it admits a Hamilton cycle. Hamiltonicity is one of the most central notions in Graph Theory, and many efforts have been devoted to obtain sufficient conditions for the existence of a Hamilton cycle (a "nice" necessary and sufficient condition should not be expected however, as deciding whether a given graph contains a Hamilton cycle is known to be NP-complete). In this chapter we will mostly concern ourselves with establishing a sufficient condition for Hamiltonicity which is applicable to a wide class of sparse graphs.

One of the first Hamiltonicity results is the celebrated theorem of Dirac [33], which asserts that if the minimum degree of a graph G on n vertices is at least $n/2$ then G is Hamiltonian. Since then, many other sufficient conditions that deal with dense graphs, were obtained (see e.g. [40] for a comprehensive reference). However, all these conditions require the graph to have $\Theta(n^2)$ edges whereas for a Hamilton cycle, only n edges are needed. Chvátal and Erdős [30] proved that if $\kappa(G) \geq \alpha(G)$ (that is, the vertex connectivity of G is at least as large as the size of a largest independent set in G) then G is Hamiltonian. Note that if G is a d-regular graph, then $\kappa(G) \leq d$ and $\alpha(G) \geq \frac{n}{d+1}$; hence the Chvátal-Erdős criterion cannot be applied if $d \leq c\sqrt{n}$ for an appropriate constant c.

When looking for sufficient conditions for the Hamiltonicity of sparse graphs, it is natural to look at random graphs with an appropriate edge probability. Erdős and Rényi [35] raised the question of what is the threshold for Hamiltonicity in random graphs. After a series of efforts by various researchers, including Korshunov [52] and Pósa [67], the problem was finally solved by Komlós and Szemerédi [53], who proved that if $p = (\log n + \log\log n + \omega(1))/n$, where $\omega(1)$ tends to infinity with n arbitrarily slowly, then $G(n,p)$ is a.s. Hamiltonian. Note that this is best possible since for $p \leq (\log n + \log\log n - \omega(1))/n$ almost surely there are vertices of degree at most one in $G(n,p)$.

The next natural step is to look for Hamilton cycles in relatively sparse pseudo-random graphs. During the last few years, several such sufficient conditions were found (see e.g. [37, 56]). These are quite complicated at times as they rely on many properties of pseudo-random graphs. Furthermore, one can argue that these conditions are not the most natural, as Hamiltonicity is a monotone increasing property, whereas pseudo-randomness is not. Our main result is a natural and simple (at least on the qualitative level) sufficient condition based on expansion and high connectivity. Before stating the result we introduce and discuss the following properties of a graph $G = (V, E)$ where $|V| = n$. As usual, the notation $N(S)$ stands for the *external neighborhood* of S, that is, $N(S) = \{v \in V \setminus S : \exists u \in S,\ (u, v) \in E\}$. Let $d = d(n)$ be a parameter.

P1 For every $S \subset V$, if $|S| \le \frac{n \log\log n \log d}{d \log n \log\log\log n}$ then $|N(S)| \ge d|S|$;

P2 There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \ge \frac{n \log\log n \log d}{4130 \log n \log\log\log n}$.

From now on, for the sake of convenience, we denote

$$m = m(n, d) = \frac{\log n \cdot \log\log\log n}{\log\log n \cdot \log d}\ .$$

Let us give an informal interpretation of the above conditions. Condition P1 guarantees *expansion*: every sufficiently small vertex subset (of size $|S| \le \frac{n}{dm}$) expands by a factor of d. Condition P2 is what can be classified as a *high connectivity* condition of some sort: every two disjoint subsets $A, B \subseteq V$ which are relatively large (of size $|A|, |B| \ge \frac{n}{4130m}$) are connected by at least one edge. Note that properties P1 and P2 together guarantee some expansion for *every* $S \subset V(G)$ of size $o(n)$. Indeed, if $|S| \le \frac{n}{dm}$ then $|N(S)| \ge d|S|$ by property P1. If $\frac{n}{dm} < |S| < \frac{n}{4130m}$ (assuming $d > 4130$) then S contains a subset of size exactly $\frac{n}{dm}$ and so by property P1 expands at least to a size of $\frac{n}{m}$, that is it expands by a factor of at least 4130. Finally, if $|S| \ge \frac{n}{4130m}$ then $N(S) \ge (1 - o(1))n$ as, by property P2, the number of vertices of $V \setminus S$ that do not have any neighbor in S is strictly less than $\frac{n}{4130m}$.

We can now state our main result:

Theorem 8.1 *Let $12 \le d \le e^{\sqrt[3]{\log n}}$ and let G be a graph on n vertices satisfying properties P1, P2 as above; then G is Hamiltonian, for sufficiently large n.*

The lower bound on d in the theorem above can probably be somewhat improved through a more careful implementation of our arguments. As for the upper bound $d \le e^{\sqrt[3]{\log n}}$, it is a mere technicality, as one expects that proving that denser graphs (that is, graphs for which d is larger) are Hamiltonian should in fact be easier. The requirement $d \le e^{\sqrt[3]{\log n}}$ makes sure (in particular) that $\frac{n}{4130m} = o(n)$ and so P2 is a non-trivial condition. We can obtain a sufficient condition for Hamiltonicity, similar to that of Theorem 8.1, and applicable to graphs with larger values of $d = d(n)$ as well; more details are given in Section 8.2.4.

It is instructive to observe that neither P1 nor P2 is enough to guarantee Hamiltonicity by itself, without relying on its companion property (unless of course they degenerate to something trivial). Indeed, for property P1 observe that the complete graph $K_{n,n+1}$ is a very strong expander locally, yet it obviously does not contain a Hamilton cycle. As for property

P2, the graph G formed by a disjoint union of a clique of size $n - \frac{n}{4130m} + 1$ and $\frac{n}{4130m} - 1$ isolated vertices clearly meets P2, but is obviously quite far from being Hamiltonian. Thus, P1 and P2 complement each other in an essential way.

Next, we discuss several applications of our main result. Theorem 8.1 was first used (see [42] and also Theorem 3.4 from Chapter 3) to address a problem of Beck [17]: it is proved that Enforcer can win the $(1, q)$ Avoider-Enforcer Hamilton cycle game, played on the edges of K_n, for every $q \leq \frac{cn \log\log\log\log n}{\log n \log\log\log n}$ where c is an appropriate constant. This was the best known bound until, very recently, Krivelevich and Szabó [58] used the same criterion 8.1 to improve this result. A similar result can be obtained for Maker in the corresponding Maker-Breaker game. The latter result falls short of the true value of the critical bias of $\frac{cn}{\log n}$ obtained by Beck (see [9]), but our proof is conceptually simpler and shorter. Moreover, Theorem 8.1 was used by Krivelevich and Szabó [58] to handle the Maker-Breaker Hamiltonicity game as well, where they improve Beck's bound.

In this chapter we prove several other corollaries of Theorem 8.1.

A graph $G = (V, E)$ is called *Hamilton-connected* if for every $u, v \in V$ there is a Hamilton path in G from u to v.

Theorem 8.2 *Let $G = (V, E)$ be a graph that satisfies properties P1 and P2; then G is Hamilton-connected.*

Remark. An immediate consequence of Theorem 8.2 is that for every edge $e \in E$ there is a Hamilton cycle of G that includes e.

A graph G is called *pancyclic* if it admits a cycle of length k for every $3 \leq k \leq n$. We prove that a graph which satisfies property P2 is "almost pancyclic".

Theorem 8.3 *Let $G = (V, E)$, where $|V| = n$ is sufficiently large, be a graph, satisfying property P2; more precisely, for every disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \geq n/t$, where $t = t(n) \geq 2$, there is an edge between a vertex of A and a vertex of B. Then G admits a cycle of length exactly k for every $\frac{8n \log n}{t \log\log n} \leq k \leq n - 3n/t$.*

Remark. The upper bound on k in Theorem 8.3 is tight up to a constant factor in the second order term, as shown by a disjoint union of $K_{n+1-n/t}$ and $n/t - 1$ isolated vertices. On the other hand, we believe that the lower bound can be improved to $\frac{c \log n}{\log t}$ for some constant c. Methods recently utilized by Verstraëte [76] and by Sudakov and Verstraëte [71] can possibly be used to establish this conjecture.

Theorem 8.1 (with minor changes to the proof) can be used to prove the following classic result (see [53]).

Theorem 8.4 *$G(n, p)$, where $p = (\log n + \log\log n + \omega(1))/n$, is a.s. Hamiltonian.*

Let $G = (V, E)$, where $|V| = n$, and let $f : \mathbb{Z}^+ \to \mathbb{R}$. A pair (A, B) of proper subsets of V is called a *separation* of G if $A \cup B = V$ and there are no edges in G between $A \setminus B$

and $B \setminus A$. The graph G is called f-connected if $|A \cap B| \geq f(|A \setminus B|)$, for every separation (A, B) of G with $|A \setminus B| \leq |B \setminus A|$. In [29] it was proved that if $f(k) \geq 2(k+1)^2$ for every $k \in \mathbb{N}$ then G is Hamiltonian for every $n \geq 3$. It was also conjectured that there exists a function f which is linear in k and is enough to ensure Hamiltonicity. Using Theorem 8.1, we can get quite close to proving this conjecture for sufficiently large n:

Theorem 8.5 *If $G = (V, E)$, where $|V| = n$, is f-connected for $f(k) = k \log k + O(1)$, then it is Hamiltonian for sufficiently large n.*

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. All of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout this chapter, log stands for the natural logarithm. We say that some event holds *almost surely*, or a.s. for brevity, if the probability it holds tends to 1 as n tends to infinity. Our graph-theoretic notation is standard and follows that of [31].

The rest of this chapter is organized as follows: in Section 8.2 we prove and discuss Theorem 8.1, in Section 8.3 we prove its corollaries: Theorems 8.2, 8.3, 8.4 and 8.5.

8.2 Proof of the main result

The proof of Theorem 8.1 is based on the ingenious rotation-extension technique, developed by Pósa [67], and applied later in a multitude of papers on Hamiltonicity (mostly of random graphs). Our proof technique borrows some technical ideas from the paper of Ajtai, Komlós and Szemerédi [1].

Before diving into fine details of the proof, we would like to compare our Hamiltonicity criterion and its proof with its predecessors. Several previous papers, including [1], [37], [56], state, explicitly or implicitly, sufficient conditions for Hamiltonicity applicable in principle to sparse graphs. Usually criteria of this sort are carefully tailored to be applied to random or pseudo-random graphs, and are therefore rather complicated and not always natural. Moreover, such criteria are sometimes fragile in the sense that they can be violated by adding more edges to the graph – a somewhat undesirable feature. Our criterion in Theorem 8.1 is (on a qualitative level, at least) quite natural and easily comprehensible, and can be potentially applied to a very wide class of graphs. As for our proof, due to the relative simplicity of the conditions we use, the argument is perhaps more involved than some of the previous proofs; there are however similarities. A novel ingredient, relying heavily on Property P2, is the part presented in Section 8.2.2 (finding many good initial rotations).

In order to be able to refer to the proof of our criterion while proving some of the corollaries we break the proof into four parts, each time indicating which property is needed for which part.

Proposition 8.6 *Let G satisfy properties P1 and P2. Then G is connected.*

Proof If not, let C be the smallest connected component of G. Then by P1, $|C| > \frac{n}{m}$, but then by P2, $E(C, V \setminus C) \neq \emptyset$ – a contradiction. $\square$

8.2.1 Constructing an initial long path

In this subsection we show that a graph which satisfies some expansion properties (that is, property P1 and some expansion of larger sets, implied by property P2) contains a long path, and even more, it has many paths of maximum length starting at the same vertex.

Let $P_0 = (v_1, v_2, \ldots, v_q)$ be a path of maximum length in G. If $1 \leq i \leq q-2$ and (v_q, v_i) is an edge of G then $P' = (v_1 v_2 \ldots v_i v_q v_{q-1} \ldots v_{i+1})$ is also of maximum length. P' is called a *rotation* of P_0 with *fixed endpoint* v_1 and *pivot* v_i. The edge (v_i, v_{i+1}) is called the *broken* edge of the rotation. We say that the segment $v_{i+1} \ldots v_q$ of P_0 is reversed in P'.

In case the new endpoint, v_{i+1}, has a neighbor v_j such that $j \notin \{i, i+2\}$, then we can rotate P' further to obtain more paths of maximum length. We use rotations and extensions together with property P1 to find a path of maximum length with large rotation endpoint sets (see for example [26], [37], [53], [56]).

Claim 8.7 *Let $G = (V, E)$ be a graph on n vertices that satisfies property P1 and moreover any subset of V of size $n/4130m$ has at least $n - o(n)$ external neighbors. Let $P_0 = (v_1, v_2, \ldots, v_q)$ be a path of maximum length in G. Then there exists a set $B(v_1) \subseteq V(P_0)$ of at least $n/3$ vertices, such that for every $v \in B(v_1)$ there is a v_1v-path of maximum length which can be obtained from P_0 by at most $\frac{2\log n}{\log d}$ rotations with fixed endpoint v_1. In particular $|V(P_0)| \geq n/3$.*

Proof Let t_0 be the smallest integer such that $\left(\frac{d}{3}\right)^{t_0-2} > \frac{n}{md}$. Note that $t_0 \leq 2\frac{\log n}{\log d}$.

We prove that there exists a sequence of sets $S_0, \ldots, S_{t_0} = B(v_1) \subseteq V(P_0) \setminus \{v_1\}$ of vertices such that for every $0 \leq t \leq t_0$, every $v \in S_t$ is the endpoint of a path, obtainable from P_0 by t rotations with fixed endpoint v_1, such that for every $0 \leq i \leq t$, after the ith rotation the non-v_1-endpoint of the path is in S_i, and moreover $|S_t| = \left(\frac{d}{3}\right)^t$ for every $t \leq t_0 - 3$, $|S_{t_0-2}| = \frac{n}{dm}$, $|S_{t_0-1}| = \frac{n}{4130m}$, and $|S_{t_0}| \geq n/3$.

First we construct the sets by induction on t. For $t = 0$, one can choose $S_0 = \{v_q\}$ and all requirements are trivially satisfied.

Induction step: let $0 < t \leq t_0 - 2$ and assume that the appropriate sets $S_0, \ldots, S_{t-1}$ with the appropriate properties were already constructed. We will now construct S_t. Let first

$$T = \{v_i \in N(S_{t-1}) : v_{i-1}, v_i, v_{i+1} \notin \bigcup_{j=0}^{t-1} S_j\} \ .$$

be the set of potential pivots for the tth rotation. Assume now that $v_i \in T$, $y \in S_{t-1}$ and $(v_i, y) \in E$. Then a v_1y-path Q can be obtained from P_0 by $t-1$ rotations such that after the jth rotation, the non-v_1-endpoint is in S_j for every $j \leq t-1$. Each such rotation breaks an edge incident with the new endpoint. Since v_{i-1}, v_i, v_{i+1} are not endpoints after any of these $t-1$ rotations, both edges (v_{i-1}, v_i) and (v_i, v_{i+1}) of the original path P_0 must be unbroken and thus must be present in Q.

Hence, rotating Q with pivot v_i will make either v_{i-1}, or v_{i+1} an endpoint (which one, depends on whether the unbroken segment $v_{i-1}v_iv_{i+1}$ is reversed or not after the first $t-1$ rotations). Assume w.l.o.g. it is v_{i-1}. We add v_{i-1} to the set $\hat{S}_t$ of new endpoints and say that v_i *placed* v_{i-1} *in* $\hat{S}_t$. The only other vertex that can place v_{i-1} in $\hat{S}_t$ is v_{i-2} (if it exists). Thus,

$$\begin{aligned} |\hat{S}_t| &\geq \frac{1}{2}|T| \geq \frac{1}{2}\left(|N(S_{t-1})| - 3(1 + |S_1| + \ldots + |S_{t-1}|)\right) \\ &\geq \frac{d}{2}\left(\frac{d}{3}\right)^{t-1} - \frac{3}{2}\frac{(d/3)^t - 1}{d/3 - 1} \geq \left(\frac{d}{3}\right)^t \end{aligned}$$

where the last inequality follows since $d \geq 12$. Clearly we can delete arbitrary elements of $\hat{S}_t$ to obtain S_t of size exactly $\left(\frac{d}{3}\right)^t$ if $t \leq t_0 - 3$ and of size exactly $\frac{n}{dm}$ if $t = t_0 - 2$. So the proof of the induction step is complete and we have constructed the sets $S_0, \ldots, S_{t_0-2}$.

To construct S_{t_0-1} and S_{t_0} we use the same technique as above, only the calculation is slightly different. Since $|N(S_{t_0-2})| \geq d \cdot \frac{n}{dm}$, we have

$$\begin{aligned} |\hat{S}_{t_0-1}| &\geq \frac{1}{2}|T| \geq \frac{1}{2}\left(|N(S_{t_0-2})| - 3(1 + |S_1| + \ldots + |S_{t_0-3}| + |S_{t_0-2}|)\right) \\ &\geq \frac{n}{2m} - \frac{3}{2}\left(\frac{(d/3)^{t_0-3} - 1}{(d/3) - 1} + 2\frac{n}{dm}\right) \geq \frac{n}{2m} - \frac{3}{2} \cdot \left(\frac{d}{3}\right)^{t_0-3} - 3\frac{n}{dm} \\ &\geq \frac{n}{2m} - \frac{3}{2} \cdot \frac{n}{dm} - 3\frac{n}{dm} \geq \frac{n}{4130m}, \end{aligned}$$

where the last inequality follows since $d \geq 12$.

For S_{t_0} the difference in the calculation comes from using the expansion guaranteed by property P2 rather than the one guaranteed by property P1, that is, $|N(S_{t_0-1})| \geq n - o(n)$. We have

$$\begin{aligned} |S_{t_0}| &\geq \frac{1}{2}|T| \geq \frac{1}{2}\left(|N(S_{t_0-1})| - 3(1 + |S_1| + \ldots + |S_{t_0-2}| + |S_{t_0-1}|)\right) \\ &\geq \frac{n}{2}(1 - o(1)) - \frac{3}{2}\left(\frac{(d/3)^{t_0-3} - 1}{(d/3) - 1} + \frac{2n}{dm} + \frac{n}{4130m}\right) \\ &\geq \frac{n}{2}(1 - o(1)) - \frac{3}{2}\left(\frac{3n}{dm} + \frac{n}{4130m}\right) \\ &\geq \frac{n}{3}, \end{aligned}$$

where the last inequality follows since $n \geq 2m$ and $d \geq 12$.

The set S_{t_0} can be chosen to be $B(v_1)$ and satisfies all the requirements of the Claim. Note that since $S_{t_0} \subseteq V(P_0)$, we have $|V(P_0)| > n/3$. This concludes the proof of the claim. □

Remark Note that, although we do not need it here, the rotations which create these paths always brake an edge of the original path P_0.

8.2.2 Finding many good initial rotations

In this subsection we prove an auxiliary lemma, which will be used in the next subsection to conclude the proof of Theorem 8.1.

Let H be a graph with a spanning path $P = (v_1, \ldots, v_l)$. For $2 \leq i < l$ let us define the auxiliary graph H_i^+ by adding a vertex and two edges to H as follows: $V(H_i^+) = V(H) \cup \{w\}$, $E(H_i^+) = E(H) \cup \{(v_l, w), (v_i, w)\}$. Let P_i be the spanning path of H_i^+ which we obtain from the path $P \cup \{(v_l, w)\}$ by rotating with pivot v_i. Note that the endpoints of P_i are v_1 and v_{i+1}.

For a vertex $v_i \in V(H)$ let S^{v_i} be the set of those vertices of $V(P) \setminus \{v_1\}$, which are endpoints of a spanning path of H_i^+ obtained from P_i by a series of rotations with fixed endpoint v_1.

A vertex $v_i \in V(P)$ is called a *bad initial pivot* (or simply a *bad vertex*) if $|S^{v_i}| < \frac{l}{43}$ and is called a *good initial pivot* (or a *good vertex*) otherwise. We can rotate P_i and find a large number of endpoints provided v_i is a good initial pivot.

Using an argument similar to the one used in the proof of Claim 8.7, we can show that H has many good initial pivots provided that a certain condition, similar to property P2, is satisfied.

Lemma 8.8 *Let H be a graph with a spanning path $P = (v_1, \ldots, v_l)$. Assume that every two disjoint sets A, B of vertices of H of sizes $|A|, |B| \geq l/43$ are connected by an edge. Then we have*

$$|R| \leq 7l/43,$$

where $R = R(P) \subseteq V(P)$ is the set of bad vertices.

Proof We will create a set $U \subseteq V(H)$, whose size is at least $|R|/7$, but does not expand enough, that is, $|U \cup N_H(U)| \leq 21|U|$. This in turn will imply that the set R of bad vertices cannot be big.

Let $R = \{v_{i_1}, \ldots, v_{i_r}\}$. We *process* the vertices of R one after the other. We will maintain subsets U and X of $V(H)$ where initially $U = X = \emptyset$. Whenever we finish processing a vertex of R we update the sets U and X. The following properties will hold after the processing of v_{i_j}.

$$U \subseteq X\,, \quad N_H(U) \subseteq ext(X)\,, \quad |U| \geq \frac{1}{7}|X|\,, \quad \{v_{i_1+1}, \ldots, v_{i_j+1}\} \subseteq X\ , \tag{8.1}$$

where $ext(X)$ denotes the set containing the vertices of X together with their left and right neighbors on P. Clearly $|ext(X)| \leq 3|X|$.

Suppose the current vertex to process is v_{i_j}. If $v_{i_j+1} \in X$, then we do not change U and X and so the conditions of (8.1) trivially hold by induction.

Otherwise, we will create sets $W_t \subseteq S^{v_{i_j}}$ inductively, such that for every t the following hold.

(a) $W_t \subseteq S_t^{v_{i_j}}$;

(b) $|W_t| = 2^t$;

(c) $W_t \cap \left(\cup_{s=0}^{t-1} W_s \cup X\right) = \emptyset$,

where $S_t^{v_{i_j}}$ contains those vertices y of $S^{v_{i_j}}$ for which a spanning path of H_i^+ ending at y can be produced from P_i by t rotations with fixed endpoint v_1, such that after the sth rotation the new endpoint is in W_s, for every $s < t$.

We begin by setting $W_0 = \{v_{i_j+1}\}$. Conditions (a) and (b) trivially hold, for condition (c) note that $v_{i_j+1} \notin X$.

Assume now that we have constructed $W_0, \ldots, W_t$ with properties $(a)-(c)$. If $|N_H(W_t) \setminus ext((\cup_{i=1}^t W_i \cup X))| > 5|W_t|$, then we create W_{t+1} with properties $(a) - (c)$, otherwise we finish the processing of v_{i_j} by updating U and X.

Let $T_t = N_H(W_t) \setminus ext((\cup_{i=1}^t W_i \cup X))$ and assume first that $|T_t| > 5|W_t|$. We use an argument similar to the one used in Claim 8.7 to create W_{t+1} with properties $(a) - (c)$.

Let $v_i \in T_t$, $v_i \neq v_1, v_l$, and suppose that v_i is adjacent to $y \in W_t$. Recall, that by property (a) a spanning path Q of H_i^+ ending at y can be produced from P_i by t rotations, such that for every $s < t$, after the sth rotation the new endpoint is in W_s. Since the vertices v_{i-1}, v_i and $v_{i+1} \notin \bigcup_{s=0}^t W_s$, they are not endpoints after any of these t rotations. Each rotation breaks an edge incident with the new endpoint, hence both edges (v_{i-1}, v_i) and (v_i, v_{i+1}) of the original path P_i must be present in Q. Rotating Q with pivot v_i will brake one of them. Such a rotation also makes one of v_{i-1} and v_{i+1} into an endpoint, and as such, into an element of $S_{t+1}^{v_{i_j}}$. Denote this vertex by v_i'. We define $W_{t+1} = \{v_i' : v_i \in T_t\}$. We say that v_i' *is placed in* W_{t+1} *by* v_i. Observe that besides v_i the only other vertex that can place v_i' in W_{t+1} is its other neighbor on the path P_i. Thus,

$$|W_{t+1}| \geq \left\lceil \frac{1}{2}(|T_t| - 2) \right\rceil \geq 2|W_t|.$$

Deleting arbitrarily some vertices from W_{t+1} we can make sure that its cardinality is exactly $2|W_t|$. Properties (a) and (b) are then naturally satisfied. Property (c) is satisfied because, by the definition of T_t we have $v_i \notin ext\,(\cup_{s=0}^t W_s \cup X)$ and so none of its neighbors on P_i, in particular v_i', is an element of $(\cup_{s=0}^t W_s \cup X)$.

Property (b) ensures that $|\cup_{s=0}^t W_s|$ is strictly increasing so the processing of the vertex v_{i_j} is bound to reach a point in which $|T_k| \leq 5|W_k|$ for some index k. At that point we update U and X by adding W_k to U and adding $W_1 \cup \dots \cup W_k \cup T_k$ to X. We have to check that the conditions of (8.1) hold.

Observe that $|W_1 \cup \dots \cup W_k| < 2|W_k|$, so the number of vertices added to X is at most seven times more than the number of vertices added to U. Also, property (c) and $U \subseteq X$ made sure that W_k was disjoint from U, so indeed the property $|U| \geq |X|/7$ remains valid. The other conditions in (8.1) follow easily from the definition of the "new" U and X. Hence the processing of v_{i_j} is complete.

Claim $|U| \leq l/43$.

Proof Assume the contrary and let j be the smallest index, such that $|U| > l/43$ after the processing of v_{i_j}.
Observe that $|U| \leq 2l/43$. Indeed, after the processing of v_{i_j} the set U received at most $|S^{v_{i_j}}|$ vertices, which is at most $l/43$, due to the fact that v_{i_j} is a bad vertex. We thus have $l/43 < |U| \leq 2l/43$, $U \subseteq X$, $N_H(U) \subseteq ext(X)$ and $|ext(X)| \leq 3|X| \leq 21|U|$. Then $|V(P) \setminus ext(X)| \geq l/43$, and there are no edges of H between U and $V(P) \setminus ext(X)$. This contradicts our assumption on H. □

To conclude the proof of the Lemma we note that after processing all vertices of R, we have $R^+ := \{v_{i_1+1}, \dots, v_{i_r+1}\} \subseteq X$ and $|U| \geq |X|/7$ by (8.1). Since $|U| \leq l/43$, it follows that $|R| = |R^+| \leq 7l/43$. □

8.2.3 Closing the maximal path

Lemma 8.9 *Let G be a connected graph that satisfies property P2. Let the conclusion of Claim 8.7 be also true for G, that is, for every path $P_0 = (v_1, v_2, \dots, v_q)$ of maximum length in G there exists a set $B(v_1) \subseteq V(P_0)$ of at least $n/3$ vertices, such that for every $v \in B(v_1)$ there is a v_1v-path of maximum length which can be obtained from P_0 by at most $t_0 \leq \frac{2\log n}{\log d}$ rotations with fixed endpoint v_1. Then G is Hamiltonian.*

Proof We will prove that there exists a path of maximum length which can be closed into a cycle. This, together with connectedness implies that the cycle is Hamiltonian. To find such a path of maximum length we will create two sets of vertices, large enough to satisfy property P2, such that between any two vertices (one from each) there is a path of maximum length.

Let $P_0 = (v_1, v_2, \dots, v_q)$ be a path of maximum length in G. Let $A_0 = B(v_1)$. For every $v \in A_0$ fix a v_1v-path $P^{(v)}$ of maximum length and, using our assumption, construct sets $B(v)$, $|B(v)| \geq n/3$, of endpoints of maximum length paths with endpoint v, obtained from a $P^{(v)}$ by at most t_0 rotations. In summary, for every $a \in A_0$, $b \in B(a)$ there is a maximum length path $P(a,b)$ joining a and b, which is obtainable from P_0 by at most $\rho := 2t_0 \leq \frac{4\log n}{\log d}$ rotations.

We consider P_0 to be directed and divided into 2ρ segments $I_1, I_2, \dots, I_{2\rho}$ of length at least $\lfloor |P_0|/2\rho \rfloor$ each, where $|P_0| \geq n/3$. As each $P(a,b)$ is obtained from P_0 by at most ρ rotations and every rotation breaks exactly one edge of P_0, the number of segments of P_0 which occur complete on this path, although perhaps reversed, is at least ρ. We say that such a segment is *unbroken*. These segments have an absolute orientation given to them by P_0, and another, relative to this one, given to them by $P(a,b)$, which we consider to be directed from a to b. We consider sequences $\sigma = I_{i_1}, I_{i_2}, \ldots, I_{i_\tau}$ of unbroken segments of P_0, which occur in this order on $P(a,b)$, where σ also specifies the relative orientation of each segment. We call such a sequence σ a *τ-sequence*, and say that $P(a,b)$ *contains* σ.

For a given τ-sequence σ, we consider the set $L(\sigma)$ of ordered pairs (a,b), $a \in A_0,\ b \in B(a)$, such that $P(a,b)$ contains σ.

The total number of τ-sequences is $2^\tau(2\rho)_\tau$. Any path $P(a,b)$ contains at least ρ unbroken segments, and thus at least $\binom{\rho}{\tau}$ τ-sequences. The average, over τ-sequences, of the number of pairs (a,b) such that $P(a,b)$ contains a given τ-sequence is therefore at least

$$\frac{n^2}{9} \cdot \frac{\binom{\rho}{\tau}}{2^\tau(2\rho)_\tau} \geq \alpha n^2,$$

where $\alpha = \alpha(\tau) = 1/9(4\tau)^{-\tau}$. Thus, there is a τ-sequence σ_0 and a set $L = L(\sigma_0)$, $|L| \geq \alpha n^2$ of pairs (a,b) such that for each $(a,b) \in L$ the path $P(a,b)$ contains σ_0. Let $\hat{A} = \{a \in P_0 : L$ contains at least $\alpha n/2$ pairs with a as first element$\}$. Then $|\hat{A}| \geq \alpha n/2$. For each $a \in \hat{A}$ let $\hat{B}(a) = \{b : (a,b) \in L\}$. Then, by the definition of $\hat{A}$, for each $a \in \hat{A}$ we have $|\hat{B}(a)| \geq \alpha n/2$.

Let $\tau = \frac{\log\log n}{2\log\log\log n}$ and let $\sigma_0 = (I_{i_1}, I_{i_2}, \ldots, I_{i_\tau})$. We divide σ_0 into two sub-sequences, $\sigma_0^1 = (I_{i_1}, \ldots, I_{i_{\tau/2}})$ and $\sigma_0^2 = (I_{i_{\tau/2+1}}, \ldots, I_{i_\tau})$ where both sub-sequences maintain the order and orientation of the segments of σ_0. Both sub-sequences σ_0^1 and σ_0^2 have at least $\tau/2 \cdot n/(6\rho) \geq \frac{n}{96m}$ vertices. Let x be the last vertex of $I_{i_{\tau/2}}$, and let y be the first vertex of $I_{i_{\tau/2+1}}$ (in the orientation given by σ_0). Now we define the notion of good vertices in σ_0^1 and σ_0^2. For σ_0^1 construct a graph H_1 from the segments of σ_0^1 by joining by an edge the last vertex of I_{i_j} to the first vertex of $I_{i_{j+1}}$ for every $1 \leq j < \tau/2$ and then adding the edges of G with both endpoints in the interior (that is, not endpoints) of segments of σ_0^1 to H_1. Then the segments of σ_0^1 with the edges linking them form an oriented spanning path in H_1, starting at x. We define *good* vertices in σ_0^1 to be the vertices which are not endpoints of any segment of σ_0^1 and are good vertices of H_1 as defined in Section 8.2.2, with $l = \sum_{j=1}^{\tau/2} |I_{i_j}|$. Due to property P2, Lemma 8.8 applies here, and so, since $\tau = o(|\sigma_0^1|)$, more than half of the vertices of σ_0^1 are good. For σ_0^2 we act similarly: construct a graph H_2 from the segments of σ_0^2 by joining the first vertex of I_{i_j} to the last vertex of $I_{i_{j-1}}$ for every $\tau/2+1 < j \leq \tau$ and then adding the edges of G with both endpoints in the interior of segments of σ_0^2 to H_2. Then the segments of σ_0^2 with the edges linking them and h_2 form an oriented spanning path in H_2, starting at y. We define *good* vertices in σ_0^2 to be the vertices which are not endpoints of any segment of σ_0^2 and are good vertices of H_2 as defined in Section 8.2.2, with $l = \sum_{j=\tau/2+1}^{\tau} |I_{i_j}|$. Due to property P2, Lemma 8.8 applies here, and so, since $\tau = o(|\sigma_0^2|)$, more than half of the vertices of σ_0^2 are good.

Since $|\hat{A}| \geq \alpha n/2 \geq \frac{n}{4130m}$ (which is why we get the upper bound on d in Theorem 8.1) and σ_0^1 has at least $|\sigma_0^1|/2 > \frac{n}{192m}$ good vertices, there is an edge from a vertex $\hat{a} \in \hat{A}$ to a good vertex in σ_0^1. Similarly, as $|\hat{B}(\hat{a})| \geq \alpha n/2$,there is an edge from some $\hat{b} \in \hat{B}(\hat{a})$ to a good vertex in σ_0^2. Consider the path $\hat{P} = P(\hat{a}, \hat{b})$ of maximum length connecting $\hat{a}$ and $\hat{b}$ and containing σ_0. The vertices x and y split this path into three sub-paths: P_1 from $\hat{a}$

to x, P_2 from y to $\hat{b}$ and P_3 from x to y. We will rotate P_1 with x as a fixed endpoint and P_2 with y as a fixed endpoint. We will show that the obtained endpoint sets V_1 and V_2 are sufficiently large. Then by property P2 there will be an edge of G between V_1 and V_2. Since we did not touch P_3, this edge closes a maximum path into a cycle, which is Hamiltonian due to the connectivity of G.

Since there is an edge from $\hat{a}$ to a good vertex in σ_0^1, by the definition of a good vertex we can rotate P_1, starting from this edge, to get a set V_1 of at least $|\sigma_0^1|/43 > n/(4130m)$ endpoints. When doing this, we will treat the subpath that links $\hat{a}$ and the first vertex of I_{i_1} and each subpath that links two consecutive segments of σ_0^1, as single edges and ignore edges of G that are incident with an endpoint of some segment of σ_0^1 - like in H_1. This ensures that all rotations and broken edges are inside segments of σ_0^1 and so there is indeed a path of the appropriate length from x to every vertex of V_1.

Similarly, since there is an edge from $\hat{b}$ to a good vertex in σ_0^2, we can rotate P_2, starting from this edge to get a set V_2 of at least $|\sigma_0^2|/43 > n/(4130m)$ endpoints. When doing this, we will treat the subpath that links $\hat{b}$ and the last vertex of I_{i_τ} and each subpath that links two consecutive segments of σ_0^2, as single edges and ignore edges of G that are incident with an endpoint of some segment of σ_0^2 - like in H_2. This ensures that all rotations and broken edges are inside segments of σ_0^2 and so there is indeed a path of the appropriate length from y to every vertex of V_2. This concludes the proof of Theorem 8.1. □

8.2.4 Hamiltonicity with larger expansion

As we have mentioned, our Hamiltonicity criterion can be extended to handle graphs with a larger expansion than that postulated in Theorem 8.1 ($d \leq e^{\sqrt[3]{\log n}}$). In particular, using very similar arguments, we can prove the following statement.

Theorem 8.10 *Let $12 \leq d \leq \sqrt{n}$ and let G be a graph on n vertices satisfying the following two properties:*

P1' *For every $S \subset V$, if $|S| \leq \frac{n \log d}{d \log n}$ then $|N(S)| \geq d|S|$;*

P2' *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \geq \frac{n \log d}{1035 \log n}$.*

Then G is Hamiltonian, for sufficiently large n.

The proof of Theorem 8.10 is almost identical to that of Theorem 8.1 given above. The only notable difference is that here we can allow ourselves to take $\tau = 2$ in the proof.

8.3 Corollaries

In this section we prove the afore-mentioned corollaries of Theorem 8.1.

Proof of Theorem 8.2 Let $G_{uv} = (V, E \cup \{(u,v)\})$; clearly G_{uv} satisfies properties P1 and P2 and is therefore Hamiltonian by Theorem 8.1. Let $C = w_1 w_2 \dots w_n w_1$ be a Hamilton cycle in G_{uv}. If (u,v) is an edge of C, remove it to obtain the desired path in G. Otherwise, assuming that $u = w_i$ and $v = w_j$, add (u,v) to $E(C)$ and remove (u, w_{i+1}) and (v, w_{j+1}), where all indices are taken modulo n, to obtain a Hamilton path of G_{uv} that contains the edge (u,v); denote this path by P. We will close P into a Hamilton cycle that includes (u,v); removing this edge will result in the required path. The building of the cycle will be done as in the proof of Theorem 8.1 Section 8.2.3, with P as P_0, while making sure that (u,v) is never broken. The proof is essentially the same, except for the following minor changes:

1. When dividing P into 2ρ segments, we will make sure that (u,v) is in one of the segments; denote it by I_j.

2. When considering τ-sequences, we will restrict ourselves to those that include I_j.

3. Assume without loss of generality that $I_j \in \sigma_0^1$. When building H_1 (and later, when rotating P_1 according to the model of H_1) we will ignore I_j, that is, we will replace it by a single edge (a,b) where a is the last vertex of I_{j-1} (or h_1 if $j = 1$) and b is the first vertex of I_{j+1} (or x if $j = \tau/2$).

□

Proof of Theorem 8.3

Fix some $\frac{8n\log n}{t\log\log n} \le k \le n - 3n/t$. Let $V_0 \subseteq V$ be an arbitrary subset of size $k + n/t$. We construct a sequence of subsets S_i, let $S_0 = \emptyset$. As long as $|S_i| < n/t$ and there exists a set $A_i \subseteq V_0 \setminus S_i$ such that $|A_i| \le n/t$ but $|N_{G[V_0 \setminus S_i]}(A_i)| < |A_i|\frac{4\log n}{\log\log n}$, we define $S_{i+1} := S_i \cup A_i$. Let q be the smallest integer such that $|S_q| \ge n/t$ or $|N_{G[V_0\setminus S_q]}(A)| \ge |A|\frac{4\log n}{\log\log n}$ for every $A \subseteq V_0 \setminus S_q$ of size at most n/t. We claim that $|S_q| < n/t$. Indeed assume for the sake of contradiction that $|S_q| \ge n/t$. Since we halt the process as soon as this occurs, and $|A_{q-1}| \le n/t$, we have $|S_q| < 2n/t$. For every $0 \le i \le q-1$ we have $|N_{G[V_0\setminus S_i]}(A_i)| < |A_i|\frac{4\log n}{\log\log n}$ and so $|N_{G[V_0]}(S_q)| < |S_q|\frac{4\log n}{\log\log n}$. On the other hand, G satisfying property P2 together with our lower bound on k implies $|N_{G[V_0]}(S_q)| > |V_0| - n/t - |S_q| \ge |V_0| - 3n/t \ge k \ge |S_q|\frac{4\log n}{\log\log n}$, a contradiction.

Hence, $|S_q| < n/t$ and so, for $U = V_0 \setminus S_q$, $G[U]$ satisfies an expansion condition similar to **P1**, that is, for every $A \subseteq U$, if $|A| \le n/t$ then $|N_{G[U]}(A)| \ge 4|A|\frac{\log n}{\log\log n}$.

In the following we prove that with positive probability the induced subgraph of G on a random k-element subset of U also satisfies a condition similar to **P1**. Let K be a k-subset of U drawn uniformly at random. We will prove that, with positive probability, $G[K]$ satisfies the following:

P1 For every $A \subseteq K$, if $|A| \le n/t$ then $|N_{G[K]}(A)| \ge 2|A|\frac{\log n}{\log\log n}$.

Let $r = |U| - k$. Note that $0 \leq r \leq n/t$. Let $A \subseteq U$ be any set of size $a \leq n/t$, then, as was noted above, $|N_{G[U]}(A)| \geq 4|A|\frac{\log n}{\log\log n}$. Let $N_0 \subseteq N_{G[U]}(A)$ be an arbitrary subset of size $4|A|\frac{\log n}{\log\log n}$. If $A \subseteq K$ and $|N_{G[K]}(A)| \leq 2|A|\frac{\log n}{\log\log n}$, then K misses at least $2|A|\frac{\log n}{\log\log n}$ vertices from N_0. This can occur with probability at most

$$\begin{aligned}
\frac{\binom{|N_0|}{\frac{2a\log n}{\log\log n}}\binom{|U|-\frac{2a\log n}{\log\log n}}{r-\frac{2a\log n}{\log\log n}}}{\binom{|U|}{r}} &\leq \binom{\frac{4a\log n}{\log\log n}}{\frac{2a\log n}{\log\log n}}\left(\frac{r}{|U|}\right)^{\frac{2a\log n}{\log\log n}} \\
&\leq 2^{\frac{4a\log n}{\log\log n}}\left(\frac{\frac{n}{t}}{\frac{8n\log n}{t\log\log n}}\right)^{\frac{2a\log n}{\log\log n}} \\
&= \left(\frac{\log\log n}{2\log n}\right)^{\frac{2a\log n}{\log\log n}}.
\end{aligned}$$

Note that the latter bound is $o(\frac{1}{n})$ for $a = 1$, and $o(\frac{1}{n}\binom{n}{a}^{-1})$ for every $a \geq 2$.

It follows by a union bound argument that

$$Pr\left[\text{there exists an } A \subseteq K \text{ such that } |A| \leq n/t \text{ but } N_{G[K]}(A) < \frac{2\log n}{\log\log n}|A|\right] = o(1).$$

Hence, there exists an k-subset X of U such that for every $A \subseteq X$, if $|A| \leq n/t$ then $|N_{G[X]}(A)| \geq \frac{2\log n}{\log\log n}|A|$. Moreover, if A, B are disjoint subsets of V, and

$$|A|, |B| \geq \frac{k\log\log k\log\left(\frac{2\log n}{\log\log n}\right)}{4130\log k\log\log\log k} \geq n/t,$$

then there is an edge between a vertex of A and a vertex of B.

Thus $G[X]$ satisfies the conditions of Theorem 8.1 with $|V| = k$ and $d = \frac{2\log n}{\log\log n}$ and is therefore Hamiltonian. It follows that G admits a cycle of length exactly k.

□

Proof of Theorem 8.4 Let $G = G(n,p) = (V,E)$ and let $d = (\log n)^{0.1}$. We begin by showing that a.s. G satisfies property P2 with respect to d. Indeed

$$\begin{aligned}
Pr[G \not\models P2] &\leq \binom{n}{\frac{n\log\log n\log d}{4130\log n\log\log\log n}}^2\left(1-\frac{\log n+\log\log n+\omega(1)}{n}\right)^{\left(\frac{n\log\log n\log d}{4130\log n\log\log\log n}\right)^2} \\
&\leq \left(\frac{4130e\log n\log\log\log n}{0.1(\log\log n)^2}\right)^{\frac{0.2n(\log\log n)^2}{4130\log n\log\log\log n}} \\
&\quad \times\exp\left\{-\frac{\log n+\log\log n+\omega(1)}{n}\cdot\frac{0.01n^2(\log\log n)^4}{4130^2(\log n)^2(\log\log\log n)^2}\right\} \\
&= o(1).
\end{aligned}$$

Next, we deal with property P1. Since a.s. there are vertices of "low" degree in G, we cannot expect every "small" set to expand by a factor of d. Therefore, to handle this difficulty, we introduce some minor changes to the proof of Theorem 8.1, in fact only to the part included in Claim 8.7. First of all, note that a.s. G is connected (this fact replaces Proposition 8.6). Let $SMALL = \{u \in V : d_G(u) \leq (\log n)^{0.2}\}$ denote the set of all vertices of G that have a "low" degree. The vertices in $SMALL$ will be called *small vertices*. Standard calculations show that a.s. G satisfies the following properties:

(1) $\delta(G) \geq 2$.

(2) For every $u \neq v \in SMALL$ we have $dist_G(u, v) \geq 250$, where $dist_G(u, v)$ is the number of edges in a shortest path between u and v in G.

(3) G satisfies a weak version of P1, that is, if $A \subseteq V \setminus SMALL$ and $|A| \leq \frac{n \log\log n \log d}{d \log n \log\log\log n}$ then $|N_G(A)| \geq 3d|A|$.

(4) The number of vertices of degree at most 11 is $O(\log^{11} n)$.

We will prove that, based on these properties, we can build initial long paths as in Claim 8.7 of the proof of Theorem 8.1; this will conclude our proof of Theorem 8.4, as in Subsections 8.2.2 and 8.2.3 we did not rely on property P1. The argument is essentially the same as in Claim 8.7; the main difference is that we will use roughly twice as many rotations to create the eventual endpoint set of size $n/3$. This extra factor two has no real effect on the rest of the proof.

Suppose first that the initial path of maximum length P_0 is such that, while creating the sets $S_1, \ldots, S_{120}$ as we did in the proof of Claim 8.7, no vertex from $\cup_{i=1}^{119} S_i$ is a small vertex. Then, by (3), like in the proof of Claim 8.7, after the ith rotation there are exactly $(3d/3)^i = (\log n)^{0.1i}$ new endpoints in S_i. Therefore, after 120 rotations we will have an endpoint set S_{120} with $(\log n)^{12}$ elements.

Suppose now that there is a vertex $u \in S_j \cap SMALL$ for some $j \leq 119$. Let P_u denote a path of maximum length from v_1 to u (which can be obtained from P_0 by at most 119 rotations). At this point we ignore the endpoint sets S_i, $i \leq j$ created so far and restart creating them. The first rotation is somewhat special. By property (1), u has at least one neighbor on P_u other than its predecessor. Thus we can rotate P_u once and obtain a v_1w-path P_w of maximum length, such that w is at distance two from a small vertex. We create new endpoint sets $S_1, \ldots, S_{120}$ with P_w as the initial path. Note that property (2) implies $w \notin SMALL$. Since a new endpoint is always at distance at most two from the old endpoint, we can rotate another 120 times without ever creating an endpoint which is a small vertex. Thus, property (3) applies and after the ith rotation (not including the one that turned w into an endpoint), $i \leq 120$, there are exactly $(3d/3)^i = (\log n)^{0.1i}$ new endpoints in S_i. Hence, after 120 further rotations we obtain a set S_{121} of size exactly $(\log n)^{12}$. Altogether we used up to 240 rotations.

In the following we will prove that the endpoint sets we build grow by the same multiplicative factor every at most *two* rotations.

We will prove by induction on t that there exist endpoint sets $S_{121}, S_{122}, \ldots$ such that for every $t \geq 122$, either $|S_t| = \frac{d}{3}|S_{t-1}|$ or $|S_t| = |S_{t-1}| = \frac{d}{3}|S_{t-2}|$.

Note that this implies $\sum_{i=0}^{t} |S_i| \leq \frac{4}{3}|S_t|$ if $|S_t| = \frac{d}{3}|S_{t-1}|$, provided n is large enough.

For the base case we just have to note that $\sum_{i=0}^{121} |S_i| \leq \frac{4}{3}|S_{121}|$. Suppose we have already built S_t for some $t \geq 121$ such that $\sum_{i=0}^{t} |S_i| \leq \frac{4}{3}|S_t|$ and now wish to build S_{t+1}. We will proceed as in the proof of Claim 8.7.

Assume first that $|N(S_t)| \geq d|S_t|$. Then, as in the proof of Claim 8.7

$$|\hat{S}_{t+1}| \geq \frac{1}{2}(d|S_t| - 3 \cdot \frac{4}{3}|S_t|) = \frac{d-4}{2}|S_t|.$$

Hence, a subset $S_{t+1} \subseteq \hat{S}_{t+1}$ with $|S_{t+1}| = \frac{d}{3}|S_t|$ can be selected.

Assume now that $|N(S_t)| < d|S_t|$. By (3), this must mean that for $S'_t := S_t \cap SMALL$ we have $|S'_t| \geq \frac{2}{3}|S_t|$. Since $|S'_t| \gg \log^{11} n$, property (4) implies that almost every vertex of S'_t has degree at least 12. By (3), no two small vertices have a common neighbor, so $|N(S'_t)| \geq (12 - o(1))|S'_t| \geq (8 - o(1))|S_t|$. As in the proof of Claim 8.7, we have

$$|\hat{S}_{t+1}| \geq \frac{1}{2}(|N(S'_t)| - 3 \cdot |\cup_{i=1}^{t} S_i|) \geq \frac{1}{2}((8 - o(1))|S_t| - 3 \cdot \frac{4}{3}|S_t|) \geq |S_t|.$$

Hence we can select an $S_{t+1} \subseteq \hat{S}_{t+1}$ such that $|S_{t+1}| = |S_t|$. Crucially, since we only used vertices from S'_t for further rotation, all the new endpoints in S_{t+1} are at distance two from a small vertex. It follows by property (2) that $S_{t+1} \cap SMALL = \emptyset$. Hence $|N(S_{t+1})| \geq 3d|S_{t+1}|$ by property (3), which implies that after the next rotation we will have

$$\hat{S}_{t+2} \geq \frac{1}{2}(3d|S_{t+1}| - 3(\frac{4}{3}|S_t| + |S_{t+1}|)) = \frac{3d-7}{2}|S_t|.$$

Hence, a subset $S_{t+2} \subseteq \hat{S}_{t+2}$ with $|S_{t+2}| = \frac{d}{3}|S_t|$ can be selected.

For the last two rotations our calculations are identical to the ones in Claim 8.7 as those depend on the expansion properties implied by condition P2.

In conclusion, we created an endpoint set $B(v_1)$ of size at least $n/3$ such that for every $v \in B(v_1)$ there is a v_1v-path of maximum length which can be obtained from P_0 by at most $240 + \frac{4 \log n}{\log d}$ rotations with fixed endpoint v_1. $\square$

Proof of Theorem 8.5

Let $G = (V, E)$ be f-connected where $f(k) = 12e^{12} + k \log k$. We prove that G satisfies conditions P1 and P2 with $d = 12$ and apply Theorem 8.1 to conclude that G is Hamiltonian for sufficiently large n. Let $A \subseteq V$ be of size at most $\frac{n}{12m}$. Either $|A| > |V \setminus (A \cup N(A))|$ and so in particular $|N(A)| \geq 12|A|$, or the pair $(A \cup N(A), V \setminus A)$ is a separation of G with $|A| \leq |V \setminus (A \cup N(A))|$ and so by our assumption $|N(A)| \geq f(|A|) \geq 12e^{12} + |A| \log |A| \geq 12|A|$. It follows that G satisfies property $P1$ with $d = 12$. Let A, B be two disjoint subsets of V such that $|B| \geq |A| \geq \frac{n}{4130m}$. Assume for the sake of contradiction that there is no edge in G between A and B; hence $(V \setminus B, V \setminus A)$ is a separation of G. By our assumption $|(V \setminus A) \cap (V \setminus B)| = |V \setminus (A \cup B)| \geq f(|A|) \geq |A| \log |A| > n$. This is clearly a contradiction and so G satisfies property $P2$ with $d = 12$. $\square$

Acknowledgements

We would like to thank Deryk Osthus for suggesting to use our criterion to address the conjecture of [29].

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