Manipulative waiters with probabilistic intuition

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Abstract

For positive integers n and q and a monotone graph property $\mathcal{A}$, we consider the two player, perfect information game $\mathrm{WC}(n, q, \mathcal{A})$, which is defined as follows. The game proceeds in rounds. In each round, the first player, called Waiter, offers the second player, called Client, $q + 1$ edges of the complete graph K_n which have not been offered previously. Client then chooses one of these edges which he keeps and the remaining q edges go back to Waiter. If at the end of the game, the graph which consists of the edges chosen by Client satisfies the property $\mathcal{A}$, then Waiter is declared the winner; otherwise Client wins the game. In this paper we study such games (also known as Picker-Chooser games) for a variety of natural graph theoretic parameters, such as the size of a largest component or the length of a longest cycle. In particular, we describe a phase transition type phenomenon which occurs when the parameter q is close to n and is reminiscent of phase transition phenomena in random graphs. Namely, we prove that if $q \leq (1-\varepsilon)n$, then Client can avoid connected components of order $c\varepsilon^{-2} \ln n$ for some absolute constant $c > 0$, whereas, for $q \geq (1+\varepsilon)n$, Waiter can force a giant, linearly sized, connected component in Client's graph. We also prove that Waiter can force Client's graph to be pancyclic for every $q \leq cn$, where $c > 0$ is an appropriate constant.

1 Introduction

The theory of positional games on graphs and hypergraphs goes back to the seminal papers of Hales and Jewett [12] and of Erdős and Selfridge [10]. It has since become a highly developed area of combinatorics (see the monograph of Beck [3] and the recent monograph [15]). The most popular and widely studied positional games are the so-called Maker-Breaker games $\mathrm{MB}(n, q, \mathcal{A})$. In each round of such games, Maker claims one previously unclaimed edge of the complete graph K_n and then Breaker responds by claiming q previously unclaimed edges. Maker's goal is to build a graph which satisfies the monotone increasing property $\mathcal{A}$, whereas Breaker aims to prevent Maker from achieving his goal. Since this is a finite, perfect information game with no chance moves and without the possibility of a draw, one of the players must have a winning strategy. It has been observed long ago, that very often (although by no means always) the winner of the game could be predicted using a heuristic known as the *probabilistic intuition*. This intuition suggests that the player who has a higher chance to win the game when both players are playing randomly is also the one who wins

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the game when both players are playing optimally. More precisely, if the random graph $G\left(n, \left\lceil \binom{n}{2}/(q+1) \right\rceil\right)$ satisfies property $\mathcal{A}$ with probability tending to 1 as n tends to infinity, then Maker has a winning strategy for $\mathrm{MB}(n, q, \mathcal{A})$. If, on the other hand, this probability tends to 0 as n tends to infinity, then $\mathrm{MB}(n, q, \mathcal{A})$ is Breaker's win. Natural examples of this fascinating phenomenon are the cases when $\mathcal{A}$ is the property of being connected [11] and when it is the property of being Hamiltonian [17].

Another class of well-studied positional games are the so-called Avoider-Enforcer games, in which Enforcer aims to force Avoider to build a graph which satisfies some monotone increasing property. Such games are sometimes referred to as misère Maker-Breaker games. There are two different sets of rules for Avoider-Enforcer games: *strict rules* under which the number of edges a player claims per round is precisely his bias and *monotone rules* under which the number of edges a player claims per round is at least as large as his bias (for more information on Avoider-Enforcer games see, for instance, [14, 13]).

In this paper we consider positional games which are closely related to Maker-Breaker and Avoider-Enforcer games; the main difference between these game types is the process of selecting edges. In every round of the *Waiter-Client* game $\mathrm{WC}(n, q, \mathcal{A})$, the first player, called Waiter, offers the second player, called Client, $q+1$ previously unclaimed edges of K_n. Client then chooses one of these edges which he keeps and the remaining q edges go back to Waiter (if in the final round of the game fewer than $q+1$ unclaimed edges remain, then all of them are claimed by Waiter). The game ends as soon as all edges of K_n have been offered. Waiter wins $\mathrm{WC}(n, q, \mathcal{A})$ if, at the end of the game, the graph consisting of all vertices of K_n and all edges claimed by Client satisfies the monotone increasing property $\mathcal{A}$; otherwise Client is the winner. We will sometimes refer to $\mathrm{WC}(n, q, \mathcal{A})$ as the $(q:1)$ Waiter-Client game $\mathcal{A}$ on $E(K_n)$. Waiter-Client games were first defined and studied by Beck under the name of Picker-Chooser (see, e.g. [2]). However, we feel that the names Waiter and Client are more suitable to describe the respective roles of the two players.

The probabilistic intuition turns out to be useful for Waiter-Client games as well. In particular, it is known to hold when $\mathcal{A}$ is the property of admitting a clique of a given fixed order [5] and when it is the property of having diameter two [8].

This article is devoted to the study of Waiter-Client games with respect to several natural *global* graph properties. In the first part of the paper we consider the order of a largest connected component in Client's graph. Playing a $(q:1)$ Waiter-Client game on $E(K_n)$, and assuming both Waiter and Client follow their optimal strategies, let $\mathcal{L}(n, q)$ denote the order of a largest connected component in Client's graph if Client tries to minimize this quantity and Waiter tries to maximize it. We prove the following phase transition type result.

Theorem 1.1. *Let n be a sufficiently large integer and let $0 < \varepsilon = \varepsilon(n) < 1$. If $q \geq (1+\varepsilon)n$, then Client has a strategy to ensure that $\mathcal{L}(n, q) \leq c\varepsilon^{-2} \ln n$ will hold for some absolute constant c. On the other hand, if $q \leq (1-\varepsilon)n$, then Waiter has a strategy to ensure that $\mathcal{L}(n, q) \geq 2\varepsilon n - 2$.*

Theorem 1.1 is a new and remarkable manifestation of the probabilistic intuition. It is well-known that, when q is close to n, the size of a largest component in Client's graph when both players play randomly (and thus Client's graph is the random graph $G(n, \lfloor \binom{n}{2}/(q+1) \rfloor))$ undergoes a phase transition (see, e.g., [7, 16]). Namely, if $q \geq (1+\varepsilon)n$, then there exists an absolute constant $c > 0$ such that, asymptotically almost surely (or a.a.s. for brevity), every connected component in RandomClient's graph has at most $c\varepsilon^{-2} \ln n$ vertices. On the other hand, if $q \leq (1-\varepsilon)n$, then a.a.s. there exists a connected component on at least $(2 + o_\varepsilon(1))\varepsilon n$ vertices in RandomClient's graph. By the aforementioned results, for every fixed $\varepsilon > 0$ and every q which is not in the *critical window* $((1-\varepsilon)n, (1+\varepsilon)n)$, the size of a largest component in Client's graph when both players play randomly and when both players follow their optimal strategies is of the same order of magnitude. Moreover, the dependency on ε exhibited by random graphs in the *sub-critical regime* and the *super-critical regime*, matches the bounds stated in Theorem 1.1.

Note that Maker-Breaker games exhibit an even stronger phase transition type behavior than that of random graphs. Indeed, it was proved in [6] that if $q = cn$ for some constant $c < 1$, then Maker can build a connected component on $\Theta(n)$ vertices, whereas, if $c > 1$, then Breaker can ensure that the order of every connected component in Maker's graph will be bounded from above by some constant (which depends on c). Moreover, it was shown in [6] that the width of the critical window is $O(\sqrt{n})$. Our next result shows that this is not the case with Waiter-Client games.

Proposition 1.2. *If $q = cn$ for some constant $c > 0$, then $\mathcal{L}(n, q) = \Omega(\ln n)$.*

Next, we consider the connectivity game, that is, the game in which Waiter's goal is to ensure $\mathcal{L}(n, q) = n$. Similarly to the Avoider-Enforcer connectivity game, played under strict rules [14], we determine the exact threshold bias for the Waiter-Client version.

Theorem 1.3. *For every integer $n \geq 4$, Waiter can force Client to build a connected subgraph of K_n if and only if $q \leq \lfloor n/2 \rfloor - 1$.*

It is interesting to note that, exactly as with strict Avoider-Enforcer games (see Theorem 1.5 in [14]), as soon as Client has $n - 1$ edges, his graph is forced to be connected. On the other hand, for Maker-Breaker games [11], monotone Avoider-Enforcer games [13] and random graphs [7, 16], connectivity requires $(1/2 + o(1))n \ln n$ edges.

In the second part of the paper we study cycle lengths in Client's graph. Our main result in this part is that soon after Waiter can force Client to build a connected graph, he can also force him to build a Hamilton cycle and, in fact, even a pancyclic graph.

Theorem 1.4. *Let n be a sufficiently large integer and let $q = q(n)$ be an integer. Then, playing a $(q : 1)$ Waiter-Client game on $E(K_n)$, the following hold:*

(i) *If $q \geq 1.1n$, then Client has a strategy to keep his graph acyclic throughout the game.*

(ii) *There exists a positive constant c such that, if $q \leq cn$, then Waiter has a strategy to ensure that, at the end of the game, Client's graph will be pancyclic.*

Let $\mathcal{C} = \mathcal{C}(n)$ denote the family of edge-sets of all connected graphs on n vertices and let $\mathcal{H} = \mathcal{H}(n)$ denote the family of all Hamiltonian graphs on n vertices. Let $q_{\mathcal{C}}$ denote the largest integer q for which Waiter has a winning strategy in $\mathrm{WC}(n, q, \mathcal{C})$ and let $q_{\mathcal{H}}$ denote the largest integer q for which Waiter has a winning strategy in $\mathrm{WC}(n, q, \mathcal{H})$. It follows from Theorems 1.3 and 1.4 that $q_{\mathcal{H}} = \Theta(q_{\mathcal{C}})$. Our next result shows that, in contrast to Maker-Breaker games, $q_{\mathcal{H}} \neq (1 + o(1))q_{\mathcal{C}}$.

Proposition 1.5. *Let n be a sufficiently large integer and let $q = q(n) \geq 0.49n$ be an integer. Then, playing a $(q : 1)$ Waiter-Client game on $E(K_n)$, Client can ensure that, at the end of the game, the minimum degree in his graph will be at most one.*

2 Preliminaries

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in some of our proofs. Most of our results are asymptotic in nature and whenever necessary we assume that the number of vertices n is sufficiently large. Our graph-theoretic notation is standard and follows that of [21]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A,B) = |E_G(A,B)|$. For a vertex $u \in V(G)$ and a set $B \subseteq V(G)$ we abbreviate $E_G(\{u\},B)$ under $E_G(u,B)$. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G which is induced on the set S. For sets $A, B \subseteq V(G)$, let $N_G(A,B) = \{v \in B \setminus A : \exists u \in A \text{ such that } uv \in E(G)\}$ denote the set of neighbors of A in $B \setminus A$. For a vertex $u \in V(G)$ and a set $B \subseteq V(G)$ we abbreviate $N_G(\{u\},B)$ under $N_G(u,B)$ and let $d_G(u,B) = |N_G(u,B)|$ denote the *degree* of u in B. We abbreviate $N_G(A,V(G))$, $N_G(u,V(G))$ and $d_G(u,V(G))$ under $N_G(A)$, $N_G(u)$ and $d_G(u)$, respectively. Often, when there is no risk of confusion, we omit the subscript G from the notation above.

Let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. The *transversal* family of $\mathcal{F}$ is $\mathcal{F}^* := \{A \subseteq X : A \cap B \neq \emptyset \text{ for every } B \in \mathcal{F}\}$.

Assume that some Waiter-Client game, played on the edge-set of some graph $H = (V,E)$, is in progress (in some of our arguments, we will consider games played on graphs other than K_n; a formal definition of such games will be given in the next paragraph). At any given moment during this game, let $G_W = (V, E_W)$ denote the graph spanned by Waiter's edges, let $G_C = (V, E_C)$ denote the graph spanned by Client's edges and let $G_F = (V, E_F)$, where $E_F = E \setminus (E_W \cup E_C)$. The edges of E_F are called *free*.

We would like to state two known game-theoretic results which will be used repeatedly in this paper. In order to do so, we need the following more general definition of Waiter-Client games. Given a positive integer q, a finite set X and a family $\mathcal{F}$ of subsets of X, the Waiter-Client game $\mathrm{WC}(X,\mathcal{F},q)$ is defined as follows. In each round, Waiter chooses $q+1$ free elements of X and offers them to Client, who then chooses one of them which he keeps and the remaining q elements are claimed by Waiter. If at some point there are less than $q+1$ free elements of X left, then Waiter claims all of them. Waiter wins $\mathrm{WC}(X,\mathcal{F},q)$ if, by the end of the game, he is able to force Client to claim all elements of some $A \in \mathcal{F}$; otherwise Client wins the game.

As observed by Beck [3], one can use the well-known potential method of Erdős and Selfridge [10], which is based on the derandomization of the first moment method, to obtain the following result.

Theorem 2.1 (implicit in [3])**.** *Let q be a positive integer, let X be a finite set, let $\mathcal{F}$ be a family of (not necessarily distinct) subsets of X and let $\Phi(\mathcal{F}) = \sum_{A \in \mathcal{F}} (q+1)^{-|A|}$. Then, playing the Waiter-Client game WC($X,\mathcal{F},q$), Client has a strategy to avoid fully claiming more than $\Phi(\mathcal{F})$ sets in $\mathcal{F}$.*

The following potential type result is a rephrased version of Corollary 1.5 from [4].

Theorem 2.2 ([4])**.** *Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|/(2q-1)} < \frac{1}{2},$$

then Waiter has a winning strategy for the WC($X,\mathcal{F}^*,q$) Waiter-Client game.

Finally, let us remark that offering more board elements in a round cannot help Waiter. Therefore, we will sometimes allow Waiter to do so. Formally, we have the following slightly stronger result.

Proposition 2.3. *Let $q \leq q'$ be positive integers, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If Waiter has a winning strategy for the WC($X,\mathcal{F},q'$) Waiter-Client game, then he also has a strategy to win the Waiter-Client game WC($X,\mathcal{F},q$) in at most $|X|/(q'+1)$ rounds.*

The rest of this paper is organized as follows: in Section 3 we prove Proposition 1.2 and Theorems 1.1 and 1.3. In Section 4 we prove Theorem 1.4 and Proposition 1.5. Finally, in Section 5 we present some open problems and conjectures.

3 The largest component

In this section we look at the size of the largest connected component Waiter can force in Client's graph. We start this study by sketching a proof of Proposition 1.2.

Proof of Proposition 1.2. We present a strategy for Waiter to force a path of length $\Theta(\ln n)$ in Client's graph. Let $t^* = \max\left\{t : \left(\frac{n}{10q}\right)^{2^t} \cdot 10q \geq q^{2/3}\right\}$; a straightforward calculation shows that $2^{t^*} = \Theta(\ln n)$. For every integer $0 \leq t \leq t^*$, Waiter will force Client to build pairwise disjoint paths $P_1, \ldots, P_{m_t}$ such that:

(a) $m_t \geq \left(\frac{n}{10q}\right)^{2^t} \cdot 10q$;

(b) $v(P_i) = 2^t$ for every $1 \leq i \leq m_t$;

(c) if $1 \leq i < j \leq m_t$, x is an endpoint of P_i and y is an endpoint of P_j, then xy is free.

In particular, $m_{t^*} \geq 1$ by definition and, as noted above, $2^{t^*} = \Theta(\ln n)$; this proves our claim.

Our claim holds trivially for $t = 0$. Assuming Waiter was able to force Client to build such paths for some integer $0 \leq t < t^*$, he will force such paths for $t+1$ as follows. For every $1 \leq i \leq m_t$, let u_i and v_i denote the endpoints of P_i. Let $X_t = \{u_1, \ldots, u_{\lfloor m_t/2 \rfloor}\}$ and let $Y_t = \{u_{\lfloor m_t/2 \rfloor+1}, \ldots, u_{m_t}\}$. Offering only edges of $E(X_t, Y_t)$, Waiter forces Client to build a matching of size at least $\left(\frac{n}{10q}\right)^{2^{t+1}} \cdot 10q$. Assuming he can do so, this ensures (a). It is easy to see that the resulting paths will satisfy Properties (b) and (c) as well. The matching is built as follows. Let $r := \left\lfloor \frac{\lfloor m_t/2 \rfloor^2 - (q+1)}{q+1+m_t} \right\rfloor$; using the definition of t^*, a straightforward calculation shows that $r \geq \left(\frac{n}{10q}\right)^{2^{t+1}} \cdot 10q$. Assume that, for some $0 \leq i \leq r-1$, Client's graph already contains a matching M_i of size i between X_t and Y_t. Waiter offers $q+1$ free edges of $E(X_t, Y_t)$ which are not incident with any vertex of $V(M_i)$. Since $(\lfloor m_t/2 \rfloor - i)(\lceil m_t/2 \rceil - i) - i(q+1) \geq q+1$ holds for every $i \leq r$, these $q+1$ edges exist and so Waiter can force the required matching in Client's graph. □

Our next goal is to prove Theorem 1.1. We consider the upper bound first. Before proving this bound, we will state and prove two simple claims.

Claim 3.1. *Let $t_\ell(k)$ denote the number of labeled spanning trees of K_k with at most ℓ leaves. If $k > 2\ell$, then $t_\ell(k) < \frac{(ek)^{2\ell} k!}{(2\ell)!}$.*

Proof. Let T be a spanning tree of K_k with at most ℓ leaves. It is easy to see that the number of vertices of T of degree at least 3 is less than ℓ. Hence, at least $k - 2\ell$ vertices of T are of degree 2. The label of each such vertex appears exactly once in the Prüfer code of T. The claim now follows since the number of appropriate sequences is at most

$$\binom{k}{k-2\ell}(k-2)_{k-2\ell}(2\ell)^{2\ell-2} < \frac{(ek)^{2\ell} k!}{(2\ell)!}.$$

□

Claim 3.2. *Let $T = (V, E)$ be a tree on m vertices. Then the number of ordered r-tuples $(P_1, \ldots, P_r)$ of (not necessarily distinct) paths of T whose union forms a subtree of T is at least $(m/4)^{2r}$.*

Proof. For every vertex $w \in V$, let $r_w = \max\{v(C) : C \text{ is a connected component of } T \setminus w\}$. Let v be a vertex which minimizes $\{r_w : w \in V\}$. Suppose for a contradiction that C_1 is a connected component of $T \setminus v$ for which $v(C_1) > m/2$. Let u denote the unique neighbor of v in C_1. It is easy to see that $r_u < r_v$ contrary to our choice of v. It follows that there exists a partition $A \cup B$ of $V \setminus \{v\}$ such that $|B| \geq |A| \geq m/4$ and $C \subseteq A$ or $C \subseteq B$ for every connected component C of $T \setminus v$. For every $(a_1, \ldots, a_r) \in A^r$ and $(b_1, \ldots, b_r) \in B^r$ there is an r-tuple $(P_1, \ldots, P_r)$ of paths of T such that a_i and b_i are the endpoints of P_i for every $1 \leq i \leq r$. Clearly, the number of such r-tuples is $|A|^r \cdot |B|^r \geq (m/4)^{2r}$. Moreover, $\bigcup_{i=1}^r P_i$ is a subtree of T since $v \in \bigcap_{i=1}^r V(P_i)$ for each such r-tuple. $\square$

We can now prove the upper bound of Theorem 1.1.

Proof. Let $\mathcal{F}_r$ be the family of all ordered r-tuples $(P_1, \ldots, P_r)$ of (not necessarily distinct) non-trivial paths of K_n for which $\bigcup_{i=1}^r P_i$ is a tree. We would like to bound $\Phi(\mathcal{F}_r) = \sum_{A \in \mathcal{F}_r} (q+1)^{-|A|}$ from above (note that there is some abuse of notation here. Formally, $\mathcal{F}_r$ is actually a multi-family of trees, where every tree T appears several times in $\mathcal{F}_r$, once for every ordered r-tuple $(P_1, \ldots, P_r)$ for which $\bigcup_{i=1}^r P_i = T$. We use this notation as we feel it will help the reader remember we are dealing with a multi-family). Let $\mathcal{F}_r^1$ denote the family of all ordered r-tuples $(P_1, \ldots, P_r) \in \mathcal{F}_r$ for which $v(\bigcup_{i=1}^r P_i) \leq 6r$ and let $\mathcal{F}_r^2$ denote the family of all ordered r-tuples $(P_1, \ldots, P_r) \in \mathcal{F}_r$ for which $v(\bigcup_{i=1}^r P_i) \geq 6r+1$. For $i \in \{1, 2\}$, let $\Phi_i = \Phi(\mathcal{F}_r^i)$; then $\Phi(\mathcal{F}_r) = \Phi_1 + \Phi_2$. We will first bound each of these terms separately.

For every $(P_1, \ldots, P_r) \in \mathcal{F}_r^1$, the tree $\bigcup_{i=1}^r P_i$ and the ordered $(2r)$-tuple of endpoints of the P_i's determine $(P_1, \ldots, P_r)$ uniquely. Hence

$$\Phi_1 \leq \sum_{k=2}^{6r} \binom{n}{k} k^{k-2} \cdot k^{2r} (q+1)^{-k+1} < n \sum_{k=2}^{6r} e^k k^{2r} \left(\frac{n}{q}\right)^{k-1} < 6rne^{6r} \cdot (6r)^{2r} < c_1^r n r^{2r+1}, \tag{1}$$

where $c_1 > 0$ is some absolute constant.

In order to obtain an effective upper bound on Φ_2 we will be more careful when estimating the number of r-tuples whose union is a given tree. If $(P_1, \ldots, P_r) \in \mathcal{F}_r^2$ and $T = \bigcup_{i=1}^r P_i$, then every leaf of T must be an endpoint of some P_i. Let $(a_1, \ldots, a_r, b_1, \ldots, b_r)$ be the vector of endpoints, where a_i and b_i are the endpoints of P_i for every $1 \leq i \leq r$. There are $(2r)_\ell$ ways to determine the leftmost position of each of these ℓ leaves in this vector and $k^{2r-\ell}$ ways to fill in the remaining $2r - \ell$ positions. Hence

$$\begin{aligned}
\Phi_2 &\leq \sum_{k=6r+1}^{n} \binom{n}{k} \sum_{\ell=2}^{2r} t_\ell(k) \cdot (2r)_\ell \cdot k^{2r-\ell} \cdot (q+1)^{-k+1} \\
&\leq \sum_{k=6r+1}^{n} \binom{n}{k} \binom{2r}{r} \sum_{\ell=2}^{2r} t_\ell(k) \cdot \ell! \cdot k^{2r-\ell} \cdot (q+1)^{-k+1} \\
&\leq q \sum_{k=6r+1}^{n} \frac{n^k}{k! q^k} \cdot 2^{2r} \cdot k^{2r} \sum_{\ell=2}^{2r} \frac{(ek)^{2\ell} k! \ell!}{(2\ell)! k^\ell} < q \sum_{k=6r+1}^{n} \frac{n^k}{q^k} \cdot 4^r e^{4r} k^{2r} \sum_{\ell=2}^{2r} \frac{k^\ell \ell!}{(2\ell)!} \\
&< q \sum_{k=6r+1}^{n} \frac{n^k}{q^k} \cdot 4^r e^{4r} k^{2r} \sum_{\ell=2}^{2r} \left(\frac{k}{\ell}\right)^\ell < q \sum_{k=6r+1}^{n} \frac{n^k}{q^k} \cdot 4^r e^{4r} k^{2r} \cdot 2r \cdot \left(\frac{k}{2r}\right)^{2r} \\
&< (1+\varepsilon) n \sum_{k=6r+1}^{n} \frac{2r \cdot e^{4r} k^{4r}}{r^{2r} (1+\varepsilon)^k},
\end{aligned} \tag{2}$$

where we used the obvious fact that the number of trees with exactly ℓ leaves is not larger than the number of trees with at most ℓ leaves in the first inequality, Claim 3.1 in the second inequality and the fact that $(k/\ell)^\ell$ is increasing for $k > 6r$ and $\ell \leq 2r$.

A straightforward calculation shows that the function $f(x) = x^{4r}(1+\varepsilon)^{-x}$ attains its maximum at $x = 4r/\ln(1+\varepsilon)$. Hence, using (2) and the fact that $\ln(1+x) \sim x$ if x tends to 0, we infer that

$$\Phi_2 \leq (1+\varepsilon)n^2 \cdot \frac{2re^{4r}(4r)^{4r}}{(\ln(1+\varepsilon))^{4r}r^{2r}} < c_2^r n^2 r^{2r+1}/\varepsilon^{4r}\,, \tag{3}$$

where $c_2 > 0$ is some absolute constant.

Combining (1) and (3), we conclude that

$$\Phi(\mathcal{F}_r) < c_3^r n^2 r^{2r+1}/\varepsilon^{4r}\,. \tag{4}$$

where $c_3 > 0$ is some absolute constant.

It thus follows by Theorem 2.1 that Client has a strategy to ensure that his graph will contain less than $c_3^r n^2 r^{2r+1}/\varepsilon^{4r}$ ordered r-tuples from $\mathcal{F}_r$. Suppose that L is a connected component of G_C of order s and let T be a spanning tree of L. In view of Claim 3.2, there are at least $(s/4)^{2r}$ ordered r-tuples of non-trivial paths of T whose union is a subtree of T. Hence

$$(s/4)^{2r} < c_3^r n^2 r^{2r+1}/\varepsilon^{4r}\,. \tag{5}$$

Substituting $r = \lfloor \ln n \rfloor$ in (5), we obtain $s \leq c\varepsilon^{-2}\ln n$, for some absolute constant $c > 0$. □

The lower bound in Theorem 1.1 is an immediate consequence of the following theorem, which will also play a crucial role in the proof of Theorem 1.3.

Theorem 3.3. *For every sufficiently large integer n and every positive integer q, Waiter has a strategy to ensure that $\mathcal{L}(n,q) \geq \min\{n, 2(n-q-1)\}$.*

Proof. Since the assertion of the theorem is trivially true for $q \geq n-2$, we can assume that $q \leq n-3$. Moreover, since $\min\{n, 2(n-q-1)\} = n$ whenever $q \leq (n-1)/2 - 1$ and since $\mathcal{L}(n,q)$ is a non-increasing function of q, we may assume that $q \geq (n-1)/2 - 1$.

We present a strategy for Waiter and then prove it allows him to ensure that Client's graph will contain a connected component on n vertices if $q = (n-1)/2 - 1$ (in particular, we assume that n is odd in this case), or on at least $2(n-q-1)$ vertices if $q \geq n/2 - 1$. At any point during the game, if Waiter is unable to follow the proposed strategy, then he forfeits the game.

In light of Proposition 2.3, we will sometimes assume that Waiter offers strictly more than $q+1$ edges in a round. The proposed strategy is divided into the following two stages.

Stage I: Waiter forces Client to build a tree T which satisfies the following three properties:

(a) $v(T) = n-q-1$.

(b) xy is free for every $x, y \in V(K_n) \setminus V(T)$.

(c) There exists an ordering $u_1, \ldots, u_{q+1}$ of the vertices of $V(K_n) \setminus V(T)$ such that $d_{G_W}(u_i, V(T)) \leq i-1$ holds for every $1 \leq i \leq \min\{n-q-1, q+1\}$ and $d_{G_W}(u_i, V(T)) \leq n-q-2$ for every $n-q \leq i \leq q+1$.

This stage lasts exactly $n-q-2$ rounds and as soon as it is over, Waiter proceeds to Stage II.

Stage II: For every $1 \leq i \leq \min\{n-q-1, q+1\}$, in the ith round of this stage Waiter offers Client all the free edges of $E(u_i, V(T) \cup \{u_1, \ldots, u_{i-1}\})$. If $q \geq n/2$, then, additionally, for every $n-q \leq j \leq q+1$, Waiter offers one arbitrary free edge of $E(u_j, V(T) \cup \{u_1, \ldots, u_{i-1}\})$.

It remains to prove that Waiter can indeed follow the proposed strategy without forfeiting the game and that, by doing so, he achieves his goal. We consider each stage separately.

Stage I: We will prove by induction on i the following more general claim: Waiter has a strategy to ensure that the following three properties will hold immediately after the ith round for every $1 \leq i \leq n-q-2$:

(a′) Client's graph is a tree T_i with i edges.

(b′) xy is free for every $x, y \in V(K_n) \setminus V(T_i)$.

(c′) There is an ordering $u_1, \ldots, u_{n-i-1}$ of the vertices of $V(K_n) \setminus V(T_i)$ such that

1. $d_{G_W}(u_j, V(T_i)) = j-1$ for every $1 \leq j \leq \min\{i+1, q+1\}$.
2. $d_{G_W}(u_j, V(T_i)) = 0$ for every $i+2 \leq j \leq n-q-1$.
3. $d_{G_W}(u_j, V(T_i)) = i$ for every $n-q \leq j \leq n-i-1$.

Note that for $i = n-q-2$, Properties (a′), (b′) and (c′) entail Properties (a), (b) and (c) (with $T = T_{n-q-2}$).

In the first round Waiter offers edges $xy_1, \ldots, xy_{q+1}$ for arbitrary vertices $x, y_1, \ldots, y_{q+1} \in V(K_n)$. Assume without loss of generality that Client selects xy_1. Clearly Properties (a′) and (b′) are satisfied. Let $z_1, \ldots, z_{n-q-2}$ be an arbitrary ordering of the vertices of $V(K_n) \setminus \{x, y_1, \ldots, y_{q+1}\}$. Taking $(u_1, \ldots, u_{n-2}) = (z_1, y_2, z_2, \ldots, z_{n-q-2}, y_3, \ldots, y_{q+1})$ shows that Property (c′) is satisfied as well. This proves our claim for $i = 1$.

Assume then that the claim holds for some $1 \leq i \leq n-q-3$; we will show that it holds for $i+1$ as well. In the $(i+1)$st round, Waiter offers all the free edges of $E(\{u_{n-q}, u_{n-q+1}, \ldots, u_{n-i-1}\} \cup \{u_{i+2}\}, V(T_i)\}$. Since $i \leq n-q-3$, it follows that $i+2 \leq n-q-1$. Hence, by the induction hypothesis, Property (c′) for i implies that the total number of edges offered is at least $((n-i-1)-(n-q)+1) \cdot 1 + 1 \cdot (i+1) = q+1$. Client selects one of these edges and it readily follows that Properties (a′) and (b′) are satisfied immediately after the $(i+1)$st round. By reordering the vertices of $V(K_n) \setminus V(T_{i+1})$ if needed, one can verify that Property (c′) is satisfied as well.

Stage II: It suffices to show that in every round of this stage, Waiter offers at least $q+1$ free edges. Let $t = 0$ if $q = (n-1)/2 - 1$ and $t = 1$ if $q \geq n/2 - 1$. Fix an arbitrary $1 \leq i \leq n-q-2+t$ and assume that Waiter did not yet forfeit the game and is about to play the ith round of Stage II (note that $q+1 = n-q-2$ for $q = (n-1)/2 - 1$). It follows by the proposed strategy for this stage, that no edge which is incident with u_i was offered by Waiter in the jth round for any $1 \leq j < i$. Therefore, we infer by Property (b) that

$$|\{u_i u_j \colon 1 \leq j < i \text{ and } u_i u_j \text{ is free}\}| = i - 1 \tag{6}$$

and by Property (c) that

$$|\{u_i w \colon w \in V(T) \text{ and } u_i w \text{ is free}\}| \geq n - q - 1 - (i-1) = n - q - i\,. \tag{7}$$

Furthermore, because of Property (c), for every $n-q \leq j \leq q+1$ there was at least one free edge in $E(u_j, V(T))$ at the end of Stage I. Similarly, because of Property (b), for every $n-q \leq j \leq q+1$ all edges of $E(u_j, \{u_1, \ldots, u_{i-1}\})$ were free at the end of Stage I. Note that, during the first $i-1$ rounds of Stage II, the number of Waiter's edges which are incident to u_j increased by at most by $i-1$. Hence, for every $1 \leq i \leq n-q-2+t$, immediately before the ith round of Stage II, there is still a free edge in $E(u_j, V(T) \cup \{u_1, \ldots, u_{i-1}\})$. It follows that

$$|\{u_j w \colon n-q \leq j \leq q+1, w \in V(T) \cup \{u_1, \ldots, u_{i-1}\} \text{ and } u_j w \text{ is free}\}| \geq 2(q+1) - n\,. \tag{8}$$

Combining (6), (7) and (8) we conclude that the number of free edges Waiter offers in the ith round of Stage II is at least $(i-1)+(n-q-i)+(2(q+1)-n) = q+1$.

Since we have shown that Waiter can play according to the proposed strategy without forfeiting the game, it readily follows from the description of the strategy that for every $1 \leq i \leq (n-q-2)+(n-q-2+t)$, immediately after the ith round, Client's graph is a tree with i edges. In particular, Client is forced to build a connected component of order $2(n-q-2)+t+1 = \min\{n, 2(n-q-1)\}$ in exactly $2(n-q-2)+t$ rounds. □

We end this section with a (by now, very easy) proof of Theorem 1.3.

Proof of Theorem 1.3. If $q \leq \lfloor n/2 \rfloor - 1$, then it follows by Theorem 3.3 that Waiter can force Client's graph to have a connected component on n vertices and thus be itself connected. On the other hand, if $q > \lfloor n/2 \rfloor - 1$, then by the end of the game, Client's graph will contain strictly less than $n-1$ edges and will therefore be disconnected. □

4 Circumference, Hamiltonicity and Pancyclicity

The main goal of this section is to prove Theorem 1.4. Starting with Part (i), we will in fact prove several results on the circumference of Client's graph. Consider a $(q:1)$ Waiter-Client game on $E(K_n)$ in which Waiter aims to maximize the circumference of Client's graph and Client tries to minimize it. Assuming both players follow their optimal strategies, denote the length of a longest cycle in Client's graph by $\mathcal{C}yc(n,q)$; if Client's graph is a forest, then we put $\mathcal{C}yc(n,q) = 0$. Our results will demonstrate that $\mathcal{C}yc(n,q)$ exhibits a similar behavior to that of the circumference of the random graph $G\left(n, \lfloor \binom{n}{2}/(q+1) \rfloor\right)$.

Theorem 4.1. *The following hold for every positive integers n and $q = q(n)$.*

(i) If $q \geq 1.1n$, then $\mathcal{C}yc(n,q) = 0$.

(ii) If $q = n + \eta$, where $1 \leq \eta = \eta(n) \leq 0.1n$, then $\mathcal{C}yc(n,q) \leq 10n/\eta \cdot \ln\ln(n/\eta)$.

(iii) If $q = n - \eta$, where $10n^{3/4} \leq \eta = \eta(n) \leq n-1$, then $\mathcal{C}yc(n,q) \geq \eta/6$.

Proof. For an integer $m \geq 3$ let $\mathcal{F}_m$ denote the family of edge-sets of all cycles of length at least m in K_n. Then

$$\begin{aligned}\Phi(\mathcal{F}_m) &= \sum_{A \in \mathcal{F}_m} (q+1)^{-|A|} = \sum_{k=m}^{n} \binom{n}{k} \frac{(k-1)!}{2} (q+1)^{-k} < \frac{1}{2} \sum_{k=m}^{\infty} \frac{1}{k} \left(\frac{n}{q}\right)^k \\ &< \frac{1}{2} \left(\frac{n}{q}\right)^{m-1} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{n}{q}\right)^k = \frac{1}{2} \left(\frac{n}{q}\right)^{m-1} \ln\left(\frac{q}{q-n}\right),\end{aligned}$$

where the last equality follows from the Taylor expansion $-\ln(1-x) = \sum_{k=1}^{\infty} x^k/k$.

Now, one can easily verify that $\Phi(\mathcal{F}_3) < 1$ if $q \geq 1.1n$ and that $\Phi(\mathcal{F}_m) < 1$ if $q = n + \eta$ for some $1 \leq \eta = \eta(n) \leq 0.1n$ and $m = \lfloor 10n/\eta \cdot \ln\ln(n/\eta) \rfloor$. Consequently, Parts (i) and (ii) of the theorem follow from Theorem 2.1.

In order to prove (iii), we present a strategy for Waiter; it is divided into the following five simple stages.

Stage I: Let $V_1 \cup V_2 \cup V_3$ be a partition of $V(K_n)$ such that $|V_1| = |V_2| = \lceil n^{3/4} \rceil$. Offering only edges of $K_n[V_3]$, Waiter forces Client to build a path $P = (u_1, \ldots, u_{\lceil \eta/2 \rceil})$.

Stage II: Let $r = \lfloor \sqrt{n} \rfloor$ and let $X_1, \ldots, X_r$ be pairwise disjoint subsets of V_1, each of size $\lfloor n^{1/4} \rfloor$. For every $1 \leq i \leq r$, in the ith round of this stage, Waiter offers Client $q+1$ arbitrary free edges of $E(X_i, \{u_1, \ldots, u_{\lfloor \eta/6 \rfloor}\})$.

Stage III: Let $Y_1, \ldots, Y_r$ be pairwise disjoint subsets of V_2, each of size $\lfloor n^{1/4} \rfloor$. For every $1 \leq i \leq r$, in the ith round of this stage, Waiter offers Client $q+1$ arbitrary free edges of $E(Y_i, \{u_{\lceil \eta/3 \rceil}, \ldots, u_{\lceil \eta/2 \rceil}\})$.

Stage IV: Let $W_1 = \{w \in V_1 : d_{G_C}(w, P) > 0\}$ and let $W_2 = \{w \in V_2 : d_{G_C}(w, P) > 0\}$. In the only round in this stage, Waiter offers Client $q+1$ arbitrary free edges of $E(W_1, W_2)$.

Stage V: Waiter offers all the remaining free edges in an arbitrary order.

It is evident that, if Waiter is able to play according to the proposed strategy, then at the end of the game, Client's graph will contain a cycle of length at least $\eta/2 - 2 \cdot \eta/6 = \eta/6$. Using our assumption that $\eta \geq 10n^{3/4}$, it is easy to verify that $e(X_i, \{u_1, \ldots, u_{\lfloor \eta/6 \rfloor}\}) \geq q+1$ and $e(Y_i, \{u_{\lceil \eta/3 \rceil}, \ldots, u_{\lceil \eta/2 \rceil}\}) \geq q+1$ hold for every $1 \leq i \leq r$ and that $e(W_1, W_2) \geq q+1$.

Hence, Waiter is able to follow Stages II, III and IV of the proposed strategy. As for Stage I, for every $1 \leq i < \eta/2$, in the ith round of the game, Waiter forces Client's graph to be a path $P_{i+1} = (u_1, \ldots, u_{i+1})$ such that every edge of $K_n[V_3 \setminus \{u_1, \ldots, u_i\}]$ is free. This is done by offering $q+1$ edges $u_i x_1, \ldots u_i x_{q+1}$ for arbitrary vertices $x_1, \ldots x_{q+1} \in V_3 \setminus \{u_1, \ldots, u_i\}$. These edges exist since $q+1 = n - \eta + 1 \leq |V_3| - i$. Finally, he can trivially follow Stage V of the proposed strategy. □

Note that in the super-critical regime, i.e. when $q < (1 - o(1))n$, our lower bound on $\mathcal{C}yc(n, q)$, stated in Theorem 4.1, is of the same order of magnitude as our lower bound on $\mathcal{L}(n, q)$, stated in Theorem 3.3. In particular, if q is such that Waiter can force Client to build a connected graph, then he can also force him to build a cycle of length $\Theta(n)$.

We would now wish to prove Theorem 1.4(ii). Before doing so, we discuss the well-known relation between expanders and Hamiltonicity. We begin by recalling some definitions and known results.

Definition 4.2. *For real numbers $d \geq 1$ and $0 < \varepsilon < 1$, a graph G is called a* (d, ε)-expander *if $|N_G(S)| \geq d|S|$ holds for every $S \subseteq V(G)$ such that $|S| \leq \varepsilon |V(G)|$.*

Definition 4.3. *For a graph G, a non-edge $uv \notin E(G)$ is called a* booster *of G, if either $G \cup \{uv\}$ is Hamiltonian, or the longest path in $G \cup \{uv\}$ is strictly longer than the longest path in G.*

The following lemma is essentially due to Pósa [20].

Lemma 4.4. *A connected non-Hamiltonian $(2, \alpha)$-expander has at least $\alpha^2 n^2/2$ boosters.*

We will deduce Theorem 1.4(ii) from a series of lemmata. Before stating our first lemma, we need one more definition.

Definition 4.5. *A bipartite graph G with bipartition (V_1, V_2) is called a* (d, ε)-half-expander *on (V_1, V_2) if $|N_G(S)| \geq d|S|$ holds for every $S \subseteq V_1$ such that $|S| \leq \varepsilon |V_1|$.*

Our first lemma asserts that, even when playing with a linear bias, Waiter can force Client to claim an edge between any two large disjoint sets.

Lemma 4.6. *Let G be a complete bipartite graph with bipartition (U, W), where $|U| = n$ and $|W| = \lfloor 5n/6 \rfloor$. Let d and $q = q(n) \leq n/(150d)$ be positive integers. Let $\mathcal{F}$ be the family of edge-sets of all induced subgraphs $G' \subseteq G$ with $V(G') = A \cup B$ such that $A \subseteq U$ is of size $a = \lceil n/(7d) \rceil$ and $B \subseteq W$ is of size $b = \lceil n/7 \rceil$. Then, for sufficiently large n, Waiter has a winning strategy in $WC(E(G), \mathcal{F}^*, q)$, i.e. he can force Client to claim at least one element in every $A \in \mathcal{F}$.*

Proof. Let q and $\mathcal{F}$ be as in the statement of the lemma and let

$$\Psi(\mathcal{F}) = \sum_{A \in \mathcal{F}} 2^{-|A|/(2q-1)}.$$

In light of Theorem 2.2, in order to prove the lemma, it suffices to verify that $\Psi(\mathcal{F}) < 1/2$. Let $m = \lfloor 5n/6 \rfloor$ and let $\delta = 1/150$. Then $q \leq \delta n/d$ and

$$\Psi(\mathcal{F}) = \binom{n}{a}\binom{m}{b} 2^{-ab/(2q-1)} \leq (20d)^{n/(7d)} \cdot 16^{n/7} \cdot 2^{-\frac{n}{7d} \cdot \frac{n}{7} \cdot \frac{d}{2\delta n}} = \left[(20d)^{1/(7d)} \cdot 16^{1/7} \cdot 2^{-1/(98\delta)}\right]^n.$$

For $\delta = 1/150$, it is easy to verify that $1/(98\delta) > 1/7 \cdot \log_2 16 + 1/(7d) \cdot \log_2(20d)$. For sufficiently large n, it then follows that

$$\Psi(\mathcal{F}) < \frac{1}{2}$$

as claimed. □

Our second lemma asserts that, playing on a complete bipartite graph, Waiter can force Client to build a half-expander.

Lemma 4.7. *Let G be a complete bipartite graph with bipartition (V_1, V_2), where $|V_1| = n$ and $|V_2| \geq n-1$. Let d and $q = q(n) \leq n/(150d)$ be positive integers. Then, for sufficiently large n, playing a $(q:1)$ Waiter-Client game on G, Waiter can force Client to build a $(d, 2/(3d))$-half-expander on (V_1, V_2).*

Proof. Let G and q be as in the statement of the lemma. Let $W \subseteq V_2$ be a set of size $\lfloor 5n/6 \rfloor$ and let $G' = G[V_1 \cup W]$. In order to prove the lemma, we present a strategy for Waiter; it is divided into the following two simple stages.

Stage I: Offering only edges of G', Waiter forces Client to build a graph $H \subseteq G'$ such that $E_H(A, B) \neq \emptyset$ for every $A \subseteq V_1$ of size $a = \lceil n/(7d) \rceil$ and every $B \subseteq W$ of size $b = \lceil n/7 \rceil$.

Stage II: At the beginning of this stage, let $\mathcal{S}$ denote the family of all inclusion minimal sets $A \subseteq V_1$ such that $|A| \leq n/(7d)$ and $|N_H(A)| < d|A|$. Let $U = \{x_1, \ldots, x_r\}$ denote the union of all sets $A \in \mathcal{S}$. At any point during this stage, let $Y = \{y \in V_2 \setminus W : d_{G_C}(y) = 0\}$. For every $1 \leq i \leq r$ and every $1 \leq j \leq d$, in round $(i-1)d + j$ of this stage, Waiter offers Client $q+1$ arbitrary free edges of $E_G(x_i, Y)$.

It follows by Lemma 4.6 that Waiter can play according to Stage I of the proposed strategy. Moreover, we claim that $|N_H(A)| \geq d|A|$ holds at the end of Stage I for every $A \subseteq V_1$ such that $n/(7d) \leq |A| \leq 2n/(3d)$. Indeed, suppose for a contradiction that there exists a set $A \subseteq V_1$ such that $n/(7d) \leq |A| \leq 2n/(3d)$ and yet $|N_H(A)| < d|A|$. Then $|W \setminus N_H(A)| > 5n/6 - d|A| \geq n/6$ but $E_H(A, W \setminus N_H(A)) = \emptyset$ contrary to Stage I of the proposed strategy.

Our next goal is to show that Waiter can play according to Stage II of the proposed strategy as well. We first claim that $r < n/(7d)$. Indeed, otherwise there exist sets $A_1, \ldots, A_t \in \mathcal{S}$ such that $n/(7d) \leq |T| \leq 2n/(7d)$, where $T := \bigcup_{i=1}^{t} A_i$. However, it follows by the definition of $\mathcal{S}$ and by a simple inductive argument that $|N_H(T)| < d|T|$ contrary to our proof that sets of such sizes expand. It now follows that $|Y| \geq |V_2 \setminus W| - dr \geq n/6 - 1 - n/7 > (q+1)d$ holds at any point during Stage II. Hence, Waiter can indeed play according to Stage II of the proposed strategy as claimed.

In order to complete the proof of our claim, that by the end of the game, Client's graph is a $(d, 2/(3d))$-half-expander on (V_1, V_2), it remains to show that small sets expand as well. Let $A \subseteq V_1$ be an arbitrary set of size $1 \leq |A| \leq n/(7d)$ such that $|N_H(A)| < d|A|$. Then there exists a partition $B \cup X_1 \cup \ldots \cup X_p$ of A (it is possible that $B = \emptyset$), where $|N_H(B)| \geq d|B|$, $p \geq 1$ and $X_i \in \mathcal{S}$ for every $1 \leq i \leq p$ (we simply successively remove inclusion minimal non-expanding subsets of A until we are left with an expanding set). It follows by Stage II of the proposed strategy that $|N_{G_C}(X_i, V_2 \setminus W)| \geq d|X_i|$ holds for every $1 \leq i \leq p$ and that $N_{G_C}(X_i, V_2 \setminus W) \cap N_{G_C}(X_j, V_2 \setminus W) = \emptyset$ holds for every $1 \leq i < j \leq p$. We conclude that $|N_{G_C}(A)| \geq |N_H(B)| + \sum_{i=1}^{p} |N_{G_C}(X_i, V_2 \setminus W)| \geq d|B| + \sum_{i=1}^{p} d|X_i| \geq d|A|$. □

Our third lemma asserts that half-expanders admit a large matching.

Lemma 4.8. *Let $d \geq 1$ and let r be a sufficiently large integer. Let X and Y be disjoint sets of sizes $|X|, |Y| \in \{r-1, r\}$ and let G be a $(d, 2/(3d))$-half-expander on (X, Y). Then, for every $T \subseteq X$ of size at most $r/(2d)$, there is a matching in G which saturates T.*

Proof. Let $T \subseteq X$ be an arbitrary set of size $t \leq r/(2d)$ and let $S \subseteq T$ be non-empty. Since $1 \leq |S| \leq r/(2d) \leq 2(r-1)/(3d)$ and since G is a $(d, 2/(3d))$-half-expander on (X, Y), it follows that $|N_G(S)| \geq |S|$. By Hall's theorem we conclude that the required matching exists. □

Our fourth lemma asserts that half-expanders admit a long path with certain additional properties.

Lemma 4.9. *Let $d \geq 2$, let r be a sufficiently large integer and let $m = \lceil 6r/5 \rceil$. For $0 \leq i \leq 3$ let X_i be a set of size $|X_i| \in \{r-1, r\}$ and let G_i be a $(d, 2/(3d))$-half-expander on (X_i, X_{i+1}), where addition is taken modulo 4. Then there exists a path $P_{m+1} = (v_0 \ldots v_m)$ in $G_0 \cup G_1 \cup G_2 \cup G_3$ such that, for every $0 \leq j \leq m$ and $0 \leq i \leq 3$, the vertex v_j is in the set X_i if and only if $j \equiv i \bmod 4$.*

Proof. Let $H = G_0 \cup G_1 \cup G_2 \cup G_3$ and let $\vec{H}$ be obtained from H by orienting an edge uv from u to v if and only if $u \in X_i$ and $v \in X_{i+1}$ for some $0 \leq i \leq 3$ (here, and throughout this proof, X_{i+1} should be read as X_0 if $i = 3$). Note that, in order to prove the lemma, it suffices to find a directed path of length m in $\vec{H}$ starting at X_0. In order to do so, we apply the DFS algorithm to $\vec{H}$ (a similar argument can be found, e.g., in [18]).

For every non-negative integer t, we denote by S^t the set of vertices of $\vec{H}$ whose exploration is complete after t steps of the algorithm, by T^t the set of vertices of $\vec{H}$ not visited thus far and put $U^t = V(\vec{H}) \setminus (S^t \cup T^t)$. Note that, for every $t \geq 0$, there are no arcs of $\vec{H}$ from S^t to T^t. Moreover, U^t spans a directed path in $\vec{H}$ for every $t \geq 0$. Therefore, it suffices to prove that there exists some $t \geq 0$ for which $|U^t| \geq m + 4$.

Since, $S^0 = \emptyset$, $T^0 = V(\vec{H})$ and, in every step of the algorithm, either $|S^t|$ is increased by 1 or $|T^t|$ is decreased by 1, it follows that there must be a step, say t_0, such that $|S^{t_0}| = |T^{t_0}|$. Hence, there exists an index $0 \leq i \leq 3$ such that

$$|S^{t_0} \cap X_i| \geq |T^{t_0} \cap X_i| \quad \text{and} \quad |S^{t_0} \cap X_{i+1}| \leq |T^{t_0} \cap X_{i+1}|\,. \tag{9}$$

Since G_i is a $(d, 2/(3d))$-half-expander on (X_i, X_{i+1}) and since $S^{t_0} \cap X_i$ has no neighbors in $T^{t_0} \cap X_{i+1}$, we infer that either

$$|S^{t_0} \cap X_i| \leq \frac{2}{3d} \cdot |X_i| \tag{10}$$

or

$$|S^{t_0} \cap X_i| > \frac{2}{3d} \cdot |X_i| \quad \text{and} \quad |T^{t_0} \cap X_{i+1}| \leq |X_{i+1}| - \frac{2}{3} \cdot |X_i|\,. \tag{11}$$

Combining (9) and (10) we obtain that

$$|U^{t_0} \cap X_i| \geq |X_i| - |S^{t_0} \cap X_i| - |T^{t_0} \cap X_i| \geq |X_i| - 2|S^{t_0} \cap X_i| \geq \left(1 - \frac{4}{3d}\right)|X_i| \geq \frac{1}{3}|X_i| \geq \frac{r-1}{3}\,. \quad (12)$$

Similarly, combining (9) and (11) we obtain that

$$|U^{t_0} \cap X_{i+1}| \geq |X_{i+1}| - |T^{t_0} \cap X_{i+1}| - |S^{t_0} \cap X_{i+1}| \geq |X_{i+1}| - 2|T^{t_0} \cap X_{i+1}| \geq \frac{4}{3} \cdot |X_i| - |X_{i+1}| \geq \frac{r-4}{3}\,. \quad (13)$$

By the definition of the arc orientations in $\vec{H}$ we have that $||U^{t_0} \cap X_i| - |U^{t_0} \cap X_j|| \leq 1$ for every $0 \leq i, j \leq 3$. Consequently, it follows by (12) and (13) that

$$|U^{t_0}| = \sum_{i=0}^{3} |U^{t_0} \cap X_i| \geq 4\left(\frac{r-4}{3} - 1\right) > \frac{6r}{5} + 4\,,$$

where the last inequality holds for sufficiently large r. □

Our fifth lemma asserts that, playing on $E(K_n)$, Waiter can quickly force Client to build a connected expander which admits a cycle of every short length.

Lemma 4.10. *Let $d \geq 4$ and let n be a sufficiently large integer. If $q \leq n/(1000d)$, then, playing a $(q:1)$ Waiter-Client game on $E(K_n)$, Waiter has a strategy to ensure that after at most $200nd$ rounds, Client's graph G_C will satisfy all of the following properties:*

(i) *G_C is a $(d/2-1, 1/(20d))$-expander;*

(ii) *G_C contains a cycle of length k for every $3 \leq k \leq \lceil n/6 \rceil$;*

(iii) *G_C is connected.*

Proof. By Proposition 2.3 we may assume that $q = \lceil n/(1000d) \rceil$. In order to prove the lemma, we present a strategy for Waiter which is divided into six stages. In the first stage Waiter will ensure Property (i). In the next four stages he will ensure Property (ii); each stage is devoted to cycles of a specific remainder modulo 4. Finally, in the last stage Waiter will ensure Property (iii).

Stage I: Let $V_1 \cup \ldots \cup V_6$ be an equipartition of $V(K_n)$. In at most $198nd$ rounds, Waiter forces Client to build $(d, 2/(3d))$-half-expanders $G_1, G_2, G_3, G_4, G_5, G_6, G_7$ on (V_1, V_2), (V_2, V_3), (V_3, V_4), (V_4, V_1), (V_1, V_5), (V_5, V_6), and (V_6, V_2), respectively.

Stage II: Let $m = \lceil n/5 \rceil$ and let $P_m = (v_1 \ldots v_m)$ be a path in $G_1 \cup G_2 \cup G_3 \cup G_4$ such that, for every $1 \leq t \leq m$ and $1 \leq r \leq 4$, the vertex v_t is in the set V_r if and only if $t \equiv r \bmod 4$. In this stage Waiter offers only edges with both endpoints in V_1. For every positive integer j such that $4j+1 \leq \lceil n/6 \rceil$, in the jth round of this stage, Waiter offers all edges of $\{v_{4i+1}v_{4i+4j+1} : 0 \leq i \leq q\}$.

Stage III: In this stage Waiter offers only edges with one endpoint in V_1 and one endpoint in V_3. For every positive integer j such that $4j-1 \leq \lceil n/6 \rceil$, in the jth round of this stage, Waiter offers all edges of $\{v_{4i+1}v_{4i+4j-1} : 0 \leq i \leq q\}$.

Stage IV: Let $\{v_{4i+1}x_{4i+1} : 0 \leq i \leq q\}$ be the edges of a matching in G_5. In this stage Waiter offers only edges with one endpoint in V_5 and one endpoint in V_3. For every positive integer j such that $4j \leq \lceil n/6 \rceil$, in the jth round of this stage, Waiter offers all edges of $\{x_{4i+1}v_{4i+4j-1} : 0 \leq i \leq q\}$.

Stage V: Let $\{x_{4i+1}y_{4i+1} : 0 \leq i \leq q\}$ be the edges of a matching in G_6. In this stage Waiter offers only edges with one endpoint in V_6 and one endpoint in V_4. For every positive integer j such that $4j+2 \leq \lceil n/6 \rceil$, in the jth round of this stage, Waiter offers all edges of $\{y_{4i+1}v_{4i+4j} : 0 \leq i \leq q\}$.

Stage VI: This stage is further divided into 5 phases. For $1 \leq i \leq 5$, in the ith phase, offering only edges of $K_n[V_{i+1}]$, Waiter forces Client to build a spanning connected subgraph of $K_n[V_{i+1}]$.

It remains to prove that Waiter can follow the proposed strategy and that, by doing so, he ensures within $200nd$ rounds that Client's graph will satisfy Properties (i), (ii) and (iii). Starting with the former, it follows from Lemma 4.7 that Waiter can play according to Stage I of the proposed strategy. The path P_m needed for Stage II exists by Lemma 4.9 and the matchings needed for Stages IV and V exist by Lemma 4.8. Since, moreover, $4q+k < m$ holds for every $k \leq \lceil n/6 \rceil$, it is straightforward to verify that Waiter can play according to Stages II, III, IV and V of the proposed strategy. Finally, since $q \leq \lfloor |V_i|/2 \rfloor - 1$ holds for every $2 \leq i \leq 6$, it follows by Theorem 1.3 that Waiter can play according to Stage VI of the proposed strategy.

Next, we prove that by following the proposed strategy, Waiter ensures that Client's graph will satisfy Properties (i), (ii) and (iii).

Let $G = (V(K_n), \bigcup_{i=1}^{7} E(G_i))$, that is, G is Client's graph at the end of Stage I. For every $1 \leq i \leq 7$, let $(V_L(G_i), V_R(G_i))$ denote the bipartition of G_i. Note that for every triple of sets (S_a, S_b, S_c) such that $1 \leq a < b < c \leq 7$ and $S_i \subseteq V_L(G_i)$ for every $i \in \{a, b, c\}$, we have $N_{G_a}(S_a) \cap N_{G_b}(S_b) \cap N_{G_c}(S_c) = \emptyset$. Since G_i is a $(d, 2/(3d))$-half-expander for every $1 \leq i \leq 7$, for every $S \subseteq V(G)$ with $|S| \leq n/(20d) < 2/(3d) \cdot \lfloor n/6 \rfloor$ we have

$$|N_G(S)| \geq \left| \bigcup_{i=1}^{7} N_{G_i}(S \cap V_L(G_i)) \right| - |S| \geq \frac{1}{2} \sum_{i=1}^{7} |N_{G_i}(S \cap V_L(G_i))| - |S| \geq \left(\frac{d}{2} - 1 \right) |S| .$$

Hence, G is $(d/2-1, 1/(20d))$-expander; this proves (i).

Fix some $3 \leq k \leq \lceil n/6 \rceil$. It is easy to verify that Waiter forced a cycle of length k in Client's graph in Stage II if $k \equiv 1 \bmod 4$, in Stage III if $k \equiv 3 \bmod 4$, in Stage IV if $k \equiv 0 \bmod 4$ and in Stage V if $k \equiv 2 \bmod 4$. This proves (ii).

In Stage VI, Waiter makes sure that $G_C[V_i]$ is connected for every $2 \leq i \leq 6$. Since, moreover, G_j is a $(d, 2/(3d))$-half-expander, for every $1 \leq j \leq 7$, it follows that G_C is connected as well. This proves (iii).

Finally, we prove that Waiter can achieve his goals quickly. It is evident that Stage I lasts at most

$$\frac{7 \cdot e(K_{\lceil n/6 \rceil, \lceil n/6 \rceil})}{q+1} \leq \frac{7 \cdot \lceil n/6 \rceil^2}{n/(1000d)} \leq 198nd$$

rounds.

In Stages II, III, IV and V, for every $3 \leq k \leq \lceil n/6 \rceil$ Waiter spends exactly one round forcing a cycle of length k in G_C. Therefore, Stages II, III, IV and V together last at most $n/6$ rounds.

It follows by Proposition 2.3 and by Theorem 1.3 that Stage VI lasts at most

$$\frac{5 \cdot e(K_{\lceil n/6 \rceil})}{\lfloor \lfloor n/6 \rfloor / 2 \rfloor} \leq (1 + o(1)) \frac{60n^2}{72n} < n$$

rounds.

To summarize, Waiter can force Client to build a graph which satisfies Properties (i), (ii) and (iii) in at most $200nd$ rounds. □

Our sixth lemma asserts that, playing on $E(K_n)$, Waiter can force Client to build a Hamiltonian expander which admits a cycle of every short length. The proof of Hamiltonicity is analogous to Beck's proof for Maker-Breaker games [1].

Lemma 4.11. *There exists a constant $c > 0$ such that for sufficiently large n and $q < cn$, playing a $(q : 1)$ Waiter-Client game on $E(K_n)$, Waiter has a strategy to force Client to build a graph G_C which satisfies all of the following properties:*

(i) *G_C is a $(2, 1/120)$-expander;*

(ii) *G_C contains a cycle of length k for every $3 \leq k \leq \lceil n/6 \rceil$;*

(iii) *G_C is Hamiltonian.*

Proof. In order to prove the lemma, we present a strategy for Waiter; it is divided into the following two stages.

Stage I: In at most $1200n$ rounds, Waiter forces Client to build a connected graph which satisfies Properties (i) and (ii).

Stage II: As long as G_C is not Hamiltonian, in every round of this stage, Waiter offers Client $q + 1$ free boosters of his current graph G_C.

Since, by definition, after claiming at most n boosters, Client's graph becomes Hamiltonian, it is evident that if Waiter can follow the proposed strategy, then he can ensure that, at the end of the game, Client's graph will satisfy Properties (i), (ii) and (iii). It thus remains to prove that he can indeed do so.

It follows by Lemma 4.10, with $d = 6$, that Waiter can play according to Stage I of the proposed strategy. As noted above, Stage II lasts at most n rounds. Therefore, the entire game lasts at most $1201n$ rounds. Since being an expander is a monotone increasing property, at any point during Stage II, G_C is a $(2, 1/120)$-expander. Therefore, by Lemma 4.4, there are at least $n^2/28800$ boosters of G_C in K_n. By choosing c to be sufficiently small, we can ensure that $n^2/28800 - 1201nq > 0$. It follows that, at any point during Stage II, there are enough free boosters for Waiter to offer. □

Finally, we can complete the proof of the main result of this section.

Proof of Theorem 1.4(ii). Let $V_1 \cup V_2$ be a partition of $V(K_n)$ such that $n_2 := |V_2| = \lfloor n/7 \rfloor$ and $n_1 := |V_1| = n - n_2$. Let $\tilde{c} < 1/120$ be the constant whose existence in ensured in Lemma 4.11, applied to K_{n_2}, and let $q < \tilde{c}n_2$. In order to prove the theorem, we present a strategy for Waiter; it is divided into the following four stages.

Stage I: Offering only edges with both endpoints in V_1, Waiter forces Client to build a graph $G_1 \subseteq K_n[V_1]$ which satisfies Properties (i), (ii) and (iii) of Lemma 4.11.

Stage II: Offering only edges with both endpoints in V_2, Waiter forces Client to build a graph $G_2 \subseteq K_n[V_2]$ which satisfies Properties (i), (ii) and (iii) of Lemma 4.11.

Stage III: Let $v_1 \ldots v_{n_1}$ be a Hamilton path in G_1. In the unique round of this stage, Waiter offers Client $q+1$ free edges of $E(v_{n_1}, V_2)$.

Stage IV: Let $v_1 \ldots v_{n_1} w_1, \ldots w_{n_2}$ be a Hamilton path in G_C and let

$$S = \{z \in V_2 : \text{ there exists a Hamilton path in } G_2 \text{ between } w_1 \text{ and } z\}.$$

For every $1 \leq j \leq n_1 - 1$, in the jth round of this stage, Waiter offers Client $q+1$ free edges of $E(v_j, S)$.

It is evident that Waiter can play according to Stage III of the proposed strategy and it follows by Lemma 4.11 that he can play according to Stages I and II as well. In order to prove that he can play according to Stage IV of the proposed strategy, it suffices to prove that $|S| \geq q+1$. It follows from Pósa's extension-rotation technique and the expansion properties of G_2 that $|S| \geq \lfloor n_2/120 \rfloor$. For sufficiently large n_2 we then have $|S| \geq \tilde{c} n_2 + 1 \geq q+1$.

Now, fix some $3 \leq k \leq n$. If $3 \leq k \leq n_1/6$, then by Lemma 4.11, there is a cycle of length k in G_1. In order to ensure the existence of long cycles, let $k = n - j + 1$ for some $1 \leq j \leq n_1 - 1$ (note that $n - n_1 + 2 = n_2 + 2 \leq n_1/6$). Let $v_j w$ denote the edge Client claims in the jth round of Stage II and let P_w be a path between w_1 and w in G_2. Then $v_j \ldots v_{n_1} w_1 P_w w v_j$ is a cycle of length k in G_C. □

We end this section with a simple proof of Proposition 1.5.

Proof of Proposition 1.5. Fix some $q \geq 0.49n$. In order to prove the theorem, we present a strategy for Client; it is divided into the following two simple stages.

Stage I: At any point during the game, let V_0 denote the set of isolated vertices in G_C and let $U = V(K_n) \setminus V_0$. As long as $|U| < n/4$, Client plays arbitrarily. As soon as $|U| \geq n/4$ first occurs, this stage is over and Client proceeds to Stage II.

Stage II: At the end of Stage I, let $x \in V_0$ be a vertex for which $d_{G_W}(x) \geq d_{G_W}(y)$ for every $y \in V_0$. In every round of this stage, if possible, Client claims an arbitrary edge which is not incident to x; otherwise, he plays arbitrarily.

It is evident that Client can follow the proposed strategy. It thus remains to prove that, by doing so, he ensures that $\delta(G_C) \leq 1$ will hold at the end of the game.

Let t denote the total number of rounds played in Stage I. At the end of Stage I, let $k = |U|$; clearly $k \in \{\lceil n/4 \rceil, \lceil n/4 \rceil + 1\}$ and $t \geq k/2$. At the end of Stage I, we have $e(G_W[U]) \leq \binom{k}{2} - e(G_C) \leq \binom{k}{2} - k/2$. Therefore, at the end of Stage I, the average over V_0 of Waiter's degree is

$$\begin{aligned} \frac{1}{n-k} \sum_{u \in V_0} d_{G_W}(u) &\geq \frac{1}{n-k} \left(tq - \binom{k}{2} + k/2 \right) \geq \frac{k(q-k+2)}{2(n-k)} \\ &\geq \frac{n(q-n/4)}{8(n-n/4)} = \frac{1}{6}\left(q - \frac{n}{4}\right) > n - 2q, \end{aligned}$$

where the last inequality holds for $q \geq 0.49n$. It thus follows by the choice of x that $d_{G_W}(x) > n - 2q$. By the description of the proposed strategy, we conclude that Client will claim at most one edge incident to x and thus $\delta(G_C) \leq d_{G_C}(x) \leq 1$ will hold at the end of the game. □

5 Concluding remarks and open problems

The giant component game. We have proved that, similarly to the random graph $G(n,p)$, the component structure of Client's graph undergoes a phase transition. Namely, we proved that if at the end of the game, Client's graph contains at most $(1-\varepsilon)n/2$ edges, where $\varepsilon > 0$ is an arbitrarily small constant, then both players have a strategy to ensure that the size of a largest connected component in Client's graph will be of order $\ln n$, whereas if Client's graph contains at least $(1+\varepsilon)n/2$ edges, then Waiter has a strategy to force a giant, linearly sized component in Client's graph. In the sub-critical regime, Client's strategy ensures that every connected component in his graph will contain at most $c\varepsilon^{-2}\ln n$ vertices for some constant $c > 0$. This is the same dependency on ε as in the random graph $G(n,(1-\varepsilon)/n)$. In the super-critical regime, Waiter's strategy ensures that the largest connected component in Client's graph will contain at least $2\varepsilon n - 2$ vertices. This is the same dependency on ε as in the random graph $G(n,(1+\varepsilon)/n)$. We believe that the latter bound (as stated in Theorem 3.3) is sharp.

Conjecture 5.1. *For any constant $\varepsilon > 0$ and sufficiently large n, if $q = (1-\varepsilon)n$, then $\mathcal{L}(n,q) = \min\{n, 2(n-q-1)\}$.*

It would be interesting to study $\mathcal{L}(n,q)$ in the critical window, i.e. when $q = (1+o(1))n$. We can prove that if $q = n+k$ and $\omega(1) = k = k(n) = o(n)$, then $\mathcal{L}(n,q) = o(n)$ (this is done by applying Theorem 2.1 to the family of labeled non-trivial paths in K_n, thus proving $\mathcal{L}(n,q) \leq 2\sqrt{n/\varepsilon}$), but there is still room for improvement. We would also like to know whether one can obtain a similar result in the super-critical regime, i.e. when $q = n-k$ for some $\omega(1) = k = k(n) = o(n)$. Another challenging task is to determine the width of the critical window.

A Waiter-Client Lehman type result. Recall that $q_{\mathcal{C}}$ denotes the largest integer q for which Waiter has a winning strategy in the $(q:1)$ Waiter-Client connectivity game on $E(K_n)$. It was proved in Theorem 1.3 that $q_{\mathcal{C}} = \lfloor n/2 \rfloor - 1$. This is **precisely** the same as the threshold bias of the strict Avoider-Enforcer connectivity game [14]. However, the arguments used for proving these two results are completely different. In particular, in [14], the result follows from a more general Lehman type result (see [19] for Lehman's Theorem). We are interested whether an analogous result holds for Waiter-Client games as well.

Question 5.2. *Let q be a positive integer and let G be a graph which admits $q+1$ pairwise edge disjoint spanning trees. Playing a $(q:1)$ Waiter-Client game on $E(G)$, is it true that Waiter can force Client to build a spanning tree?*

For $q = 1$, this question was answered affirmatively in [9].

Avoiding cycles. It was proved in Theorem 4.1 that, for $q \geq 1.1n$, Client can keep his graph acyclic, whereas for $q \leq (1-\varepsilon)n$, Waiter can force Client to build a cycle. We believe that the latter is asymptotically tight.

Conjecture 5.3. *For any constant $\varepsilon > 0$ and integer $q \geq (1+\varepsilon)n$, playing a $(q:1)$ Waiter-Client game on $E(K_n)$, Client has a strategy to keep his graph acyclic.*

Note that, similarly to the case of $\mathcal{L}(n,q)$ we know very little about the behavior of $\mathcal{C}yc(n,q)$ in the critical window, i.e. when $q = (1+o(1))n$.

Minimum degree k and k-connectivity. We determined $q_{\mathcal{C}}$ precisely in Theorem 1.3 and determined $q_{\mathcal{H}}$ up to a multiplicative constant factor in Theorem 1.4. There are other natural graph properties $\mathcal{A}$, which seem simpler than Hamiltonicity, for which we can prove that $q_{\mathcal{A}} = \Theta(n)$ but cannot determine its asymptotic value (as for the special cases of connectivity and Hamiltonicity, $q_{\mathcal{A}}$ denotes the largest integer q for which Waiter has a winning strategy in $\mathrm{WC}(n, q, \mathcal{A})$). For example, for a positive integer k, let $\mathcal{D}_k = \mathcal{D}_k(n)$ denote the property of n-vertex graphs having minimum degree at least k and let $\mathcal{C}_k = \mathcal{C}_k(n)$ denote the property of being k-vertex-connected. It is not hard to see that $n/(2k) - 3 \leq q_{\mathcal{C}_k} \leq q_{\mathcal{D}_k} \leq n/k$. Indeed, the upper bound follows directly from the simple fact that every graph on n vertices with minimum degree at least k has at least $kn/2$ edges; we can in fact improve it slightly by an argument analogous to the proof of Proposition 1.5. The lower bound can be obtained via the following simple strategy. Let $V_1 \cup \ldots \cup V_k$ be an equipartition of $V(K_n)$. Using Theorem 1.3, Waiter first forces Client to build a connected graph on V_i for every $1 \leq i \leq k$. For every $1 \leq i < j \leq k$, let $G_{ij} = (V_i \cup V_j, E(V_i, V_j))$ and let H_{ij}^1 and H_{ij}^2 be edge disjoint $(q+1)$-regular subgraphs of G_{ij}. In an arbitrary order, for every $1 \leq i < j \leq k$, every $x \in V_i$ and every $y \in V_j$, Waiter offers $q+1$ edges of H_{ij}^1 incident to x and $q+1$ edges of H_{ij}^2 incident to y. It is not hard to see that the graph built by Client admits k internally pairwise vertex disjoint paths between any pair of vertices and is thus k-connected by Menger's Theorem. We are currently not able to determine $q_{\mathcal{D}_k}$ asymptotically; not even for $k = 1$. At present, we are also unable to determine $q_{\mathcal{C}_k}$ asymptotically for any $k \geq 2$.

Similarly, we do not know the answer to the following two questions (though we suspect it is positive):

Question 5.4. *Is there an integer $k \geq 2$ for which $q_{\mathcal{D}_k}$ is substantially larger than $q_{\mathcal{C}_k}$?*

Question 5.5. *Is $q_{\mathcal{C}_2}$ substantially larger than $q_{\mathcal{H}}$?*

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References

[1] J. Beck, *Random graphs and positional games on the complete graph*, Annals of Discrete Mathematics **28** (1985), 7–13.

[2] J. Beck, *Positional games and the second moment method*, Combinatorica **22** (2002), 169–216.

[3] J. Beck, **Combinatorial games: Tic-tac-toe theory**, Encyclopedia of Mathematics and its Applications, **114**. Cambridge University Press, Cambridge, 2008. xiv+732 pp.

[4] M. Bednarska-Bzdęga, *On weight function methods in Chooser-Picker games*, Theoretical Computer Science **475** (2013), 21–33.

[5] M. Bednarska-Bzdęga, D. Hefetz and T. Łuczak, *Picker-Chooser fixed graph games*, arXiv:1402.7308.

[6] M. Bednarska and T. Łuczak, *Biased positional games and the phase transition*, Random Structures and Algorithms **18** (2001) 141–152.

[7] B. Bollobás, **Random Graphs**, Academic Press, London, 1985.

[8] A. Csernenszky, *The Picker-Chooser diameter game*, Theoretical Computer Science **411** (2010), 3757–3762.

[9] A. Csernenszky, C. I. Mándity and A. Pluhár, *On Chooser-Picker positional games*, Discrete Mathematics **309** (2009), 5141–5146.

[10] P. Erdős and J.L. Selfridge, *On a combinatorial game*, Journal of Combinatorial Theory Series A **14** (1973), 298–301.

[11] H. Gebauer and T. Szabó, *Asymptotic random graph intuition for the biased connectivity game*, Random Structures and Algorithms **35** (2009), 431–443.

[12] A. W. Hales and R. I. Jewett, *Regularity and positional games*, Transactions of the American Mathematical Society **106** (1963), 222–229.

[13] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, *Avoider-Enforcer: The rules of the game*, Journal of Combinatorial Theory Series A **117** (2010), 152–163.

[14] D. Hefetz, M. Krivelevich and T. Szabó, *Avoider-Enforcer games*, Journal of Combinatorial Theory Series A **114** (2007), 840–853.

[15] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, **Positional Games**, Birkhäuser, 2014.

[16] S. Janson, T. Łuczak and A. Ruciński, **Random graphs**, Wiley, 2000.

[17] M. Krivelevich, *The critical bias for the Hamiltonicity game is* $(1 + o(1))n/\ln n$, Journal of the American Mathematical Society **24** (2011), 125–131.

[18] M. Krivelevich and B. Sudakov, *The phase transition in random graphs – a simple proof*, Random Structures and Algorithms **43** (2013), 131–138.

[19] A. Lehman, *A solution of the Shannon switching game*, J. Soc. Indust. Appl. Math. **12** (1964), 687–725.

[20] L. Pósa, *Hamiltonian circuits in random graphs*, Discrete Mathematics **14** (1976), 359–364.

[21] D. B. West, **Introduction to Graph Theory**, Prentice Hall, 2001.