On antimagic directed graphs

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July 3, 2008

Abstract

An antimagic labeling of an undirected graph G with n vertices and m edges is a bijection from the set of edges of G to the integers $\{1, \ldots, m\}$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called *antimagic* if it admits an antimagic labeling. In [5], Ringel has conjectured that every simple connected graph, other than K_2, is antimagic. Despite some effort in recent years, this conjecture is still open. In this paper we consider a natural variation; namely, we consider antimagic labelings of directed graphs. In particular, we prove that every digraph, whose underlying undirected graph is "dense" is antimagic, and that almost every undirected d-regular graph admits an orientation which is antimagic.

1 Introduction

An antimagic labeling of an undirected graph G with n vertices and m edges is a bijection from the set of edges of G to the integers $\{1, \ldots, m\}$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called *antimagic* if it admits an antimagic labeling. In [5], Ringel has conjectured that every simple connected graph, other than K_2, is antimagic. This conjecture is still open, however, some special cases of it were proved. Alon et al [1] proved that large dense graphs are antimagic; that is, there exists an absolute constant $C > 0$ such that any graph with n vertices and minimum degree at least $C \log n$ is antimagic. Hefetz [6] proved that

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any graph on $n = 3^k$ vertices that admits a triangle factor is antimagic. Cranston [2] proved that any regular bipartite graph with minimum degree at least 2 is antimagic. For more related results the reader is referred to the survey [4].

One can define antimagic digraphs in the obvious analogous way. An antimagic labeling of a *directed* graph D with n vertices and m arcs is a bijection from the set of arcs of D to the integers $\{1, \ldots, m\}$ such that all n *oriented vertex sums* are pairwise distinct, where an oriented vertex sum is the sum of labels of all edges entering that vertex minus the sum of labels of all edges leaving it. A digraph is called *antimagic* if it admits an antimagic labeling. A natural question to ask would be: "Is every simple connected digraph antimagic?" Note that the answer is positive for one directed edge. We would like to rephrase this question in the following way: "Is every orientation of any simple connected undirected graph antimagic?" Relaxing this form, we obtain the following question: "Given any undirected graph G, does there exist an orientation D of G which is antimagic?" For a given graph G, we call such an orientation *an antimagic orientation of* G. In this paper we will provide partial answers to both questions.

Starting with the former, it is possible to adapt the proof of Alon et al [1] of the aforementioned result to directed graphs (though this is not entirely straightforward as some technical problems arise). This provides an affirmative answer for the first question for "dense" graphs.

Theorem 1.1 *There exists an absolute constant C such that the following holds. Let G be any undirected graph on n vertices, with minimum degree at least $C \log n$; then, every orientation of G is antimagic.*

It is easy to see that, in general, the answer to the first question is "no". Indeed, both $K_{1,2}$ and K_3 have an orientation which is not antimagic (orient it such that the out-degree and in-degree of every vertex is at most 1), so not every directed graph is antimagic. However, these are the only (connected) counterexamples we have found. It is therefore still possible that every other directed graph, whose underlying undirected graph is connected, is antimagic. Except for Theorem 1.1 our results in this setting are much more modest.

Theorem 1.2 *For every orientation of every undirected graph that belongs to one of the following families, there exists an antimagic labeling.*

1. *Stars S_n on $n+1$ vertices for every $n \neq 2$.*
2. *Wheels W_n on $n+1$ vertices for every $n \geq 3$.*
3. *Cliques K_n on n vertices for every $n \neq 3$.*

The second question is easier to tackle. First, it is clear that the results of Theorems 1.1 and 1.2 also hold in this setting, as the second question is a relaxation of the first. Moreover, it is easy to see that every bipartite antimagic undirected graph $G = (A \uplus B, E)$ admits an antimagic orientation. Indeed, one can direct all edges from A to B and apply any of the antimagic labelings of G (clearly, this changes only the signs of the vertex sums of the vertices of A). Using this observation and the aforementioned result of Cranston [2], we immediately conclude that every regular bipartite graph admits an antimagic orientation. We generalize this result as follows:

Theorem 1.3 *Let $G = (V, E)$ be a $(2d + 1)$-regular (not necessarily connected) undirected graph with $d \geq 0$; then there exists an antimagic orientation of G.*

Theorem 1.4 *Let $G = (V, E)$ be a $2d$-regular undirected connected graph with $d \geq 1$. If G admits a matching that covers all but at most one vertex, then there exists an antimagic orientation of G.*

Remark It seems hard to discard any of the two conditions in Theorem 1.4, that is connectedness and having a matching that covers all vertices but at most one. In fact, we do not even know if every 2-factor admits an antimagic orientation.

Since for every $d \geq 3$ a random d-regular graph on n vertices almost surely is connected and admits a matching that covers all but at most one vertex, we immediately obtain the following corollary:

Corollary 1.5 *For every $d \geq 3$ and sufficiently large n, almost every d-regular graph on n vertices has an antimagic orientation.*

A different approach to generalizing (a directed version of) Cranston's result [2], is manifested in the following theorem and its immediate corollary.

Theorem 1.6 *Let G be a graph on $2n$ vertices that admits a perfect matching, and let U be an independent set in G of size n. If $\deg(u) \geq 3$ for every $u \in U$, then G admits an antimagic orientation.*

Corollary 1.7 *Let $G = (A \uplus B, E)$ be a bipartite graph that admits a perfect matching. If $\deg(a) \geq 3$ for every $a \in A$ or $\deg(b) \geq 3$ for every $b \in B$, then G admits an antimagic orientation.*

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. Some of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [3].

The rest of this paper is organized as follows: In Section 2 we prove Theorem 1.1 and Theorem 1.2. In Section 3 we prove Theorems 1.3, 1.4 and 1.6. In Section 4 we present some conclusions and open problems.

2 Graphs for which every orientation is antimagic

In this section we prove Theorem 1.1 and Theorem 1.2. Our proof of Theorem 1.1 follows that of Theorem 1.2 from [1], but it also contains some new ideas. We will omit certain details if they are very similar to the ones from [1], and elaborate on the different ones. The main differences between both proofs are in Lemma 2.2 and in phase 5 of the proof of the theorem.

For a graph $G = (V, E)$ and a vertex $u \in V$, let $\hat{\Gamma}(u) := \{(u, v) : v \in \Gamma(u)\}$ denote the set of edges which are incident with u in G. If the context is clear we will write $\Gamma(v)$ and $\hat{\Gamma}(v)$ instead of $\Gamma_G(v)$ and $\hat{\Gamma}(v)$ respectively.

Lemma 2.1 *There exist absolute positive constants C_1, C_2 such that the following holds. Let t and d be integers such that $C_1 \log t \leq d \leq t/30$, and for $1 \leq i \leq d$ let $a_{i,1}, a_{i,2}$ be $2d$ distinct elements of $\{1, \dots, t\}$, chosen randomly with all choices of d pairwise disjoint pairs being equally likely. Let $\sigma \in \{-1, +1\}^d$ be any sign vector. Then, with probability at least $1 - 1/t^2$ the following holds. For every $1 \leq i \leq d$, choose $j_i \in \{1, 2\}$ randomly, independently and uniformly, and consider the random sum $Q = \sum_{i=1}^{d} \sigma_i a_{i,j_i}$. Then, for every integer s, the probability that Q is equal to s is at most $\frac{C_2}{t\sqrt{d}}$.*

We omit the proof, as it is a straightforward adaption of the proof of Lemma 2.3 from [1].

Lemma 2.2 *Let C be a positive constant and let $d = \lfloor C \log(n) \rfloor$. For sufficiently large n the following holds. Let $G = (A \uplus B, E)$ be a graph with n vertices and $t \in \mathbb{N}_{\text{even}}$ edges, such that $\deg(v) \in \{d - 1, d\}$ for every $v \in A$, $\deg(v) \geq d + 1$ for every $v \in B$, and B is an independent set. Moreover, let $D(G)$ be any orientation of G.*
Then there exists a function $F : B \to 2^E$ that assigns to each $v \in B$ a subset of the edges that are incident with v, and a bijective mapping $p : E \to E$ (a pairing of the edges of G) such that the following properties hold:

(1) For every $v \in B$, the set $F(v)$ contains an even number of edges; moreover

$$d + 1 \leq \deg(v) - |F(v)| \leq 2d.$$

(2) For every $v \in B$, we have $p(F(v)) = F(v)$. Moreover, if an edge $e \in \hat{\Gamma}(v)$ is directed away from v in $D(G)$, then $p(e)$ is also directed away from v, while if e is directed towards v in $D(G)$, then $p(e)$ is also directed towards v.

(3) If $e \in E \setminus F(B)$, where $F(B) := \bigcup_{v \in B} F(v)$, then e and $p(e)$ are independent edges (that is edges that do not share a vertex).

(4) For any two vertices $u, v \in V$, we have $|p(\hat{\Gamma}(u) \setminus F(u)) \cap \hat{\Gamma}(v)| \leq 100$ (if $u \in A$, then we use the convention $F(u) = \emptyset$).

Proof We first describe how to define F and $p|_{F(B)}$ successively, such that (1), (2) and (4) are satisfied.

Initially, $F(v) = \emptyset$ for every $v \in B$. In each step, pick a vertex $v \in B$ with $\deg(v) - |F(v)| \geq 2d + 1$ (if no such vertex exists we are done). Partition the edges of $\hat{\Gamma}(v) \setminus F(v)$ into two sets, those that are directed away from v in $D(G)$ and those that are directed towards v. Let T be the larger of the two sets (breaking ties arbitrarily). Note that $|T| \geq d + 1$. Choose two edges $(v, x), (v, y)$ from T, for which there is no other vertex $w \in B$ such that $(w, x), (w, y) \in F(w)$ and $p((w, x)) = (w, y)$ (this constraint ensures that property (4) holds with a stronger bound of 1 on the right hand side of the inequality). Such a choice is possible, since, if for every two edges $(v, x), (v, y)$ from T, there was a vertex $w \in B$ such that $(w, x), (w, y) \in F(w)$ and $p((w, x)) = (w, y)$, then the vertices in $\Gamma(v) \cap V(T) \subseteq A$ would be of a degree which is strictly larger than d; clearly, this is a contradiction. Add (v, x) and (v, y) to $F(v)$, and set $p((v, x)) = (v, y)$ and $p((v, y)) = (v, x)$.

Next we will show how to extend p to all elements of E, without affecting (1) and (2), such that (3) and (4) are satisfied.

Let $G' = (V(G), E \setminus F(B))$, and let k denote the number of edges of G'. Note that $k = t - |F(B)|$ is even and that $k \geq \max(t - |B|(n-1), |B|(d+1)) \geq d^2/3$ for sufficiently large n (for the last estimate we use the bound $t \geq (d-1)n/2$). Let $L(G')$ denote the line graph of G', and let $\overline{L(G')}$ denote its complement. Since $\Delta(G') \leq 2d$, it follows that $\delta(\overline{L(G')}) \geq k - 1 - 2(2d-1) \geq \frac{k}{2} = \frac{1}{2}|V(\overline{L(G')})|$ holds for sufficiently large n. We can thus apply Dirac's theorem to conclude that there exists a Hamilton cycle in $\overline{L(G')}$, and in particular, also a perfect matching (recall that k is even). This immediately defines an extension of the mapping p to the edges of G' such that, for every edge $e \in E(G')$, e and $p(e)$ are independent.

So far the functions F and p satisfy (1), (2) and (3), but condition (4) might be violated by our choice of the perfect matching in $\overline{L(G')}$. Hence, in order to complete the proof, it suffices to show how one can modify $p|_{E(G')}$, such that the number of pairs of vertices $u, v \in V$, for which $|\{\{e^*, p(e^*)\} : e^* \in \hat{\Gamma}_{G'}(u) \wedge p(e^*) \in \hat{\Gamma}_{G'}(v)\}| \geq 100$ can be reduced by one (if for all pairs of vertices the cardinality of this set is bounded by 99, then, together with the contribution of $+1$ we get from the definition of F at the beginning of the proof, we can ensure that the inequality in condition (4) holds). Let u be a vertex and let $e \in \hat{\Gamma}_{G'}(u)$

be an edge, such that $e' := p(e)$ shares an endpoint v with at least 99 more edges from $p(\hat{\Gamma}_{G'}(u))$. Let P_{uv} denote the set of edge pairs $\{\{e^*, p(e^*)\} : e^* \in E(G')\}$ that share an endpoint with e or with e'; clearly, $|P_{uv}| \leq 8d$. For every vertex $z \in V(G')$, let S_z denote the set of all vertices that have a degree of at least 99 in the graph spanned by the edges of $p(\hat{\Gamma}_{G'}(z))$. Note that for every $z \in V(G')$, we have $|p(\hat{\Gamma}_{G'}(z))| \leq 2d$ and thus $|S_z| \leq 4d/99$. Let Q_{uv} be the set of edge pairs $\{\{e^*, p(e^*)\} : e^* \in E(G')\}$ for which either e^* or $p(e^*)$ has an endpoint in S_u or in S_v. Note that $|Q_{uv}| \leq 2d(|S_u| + |S_v|) \leq 16d^2/99$. Pick an arbitrary edge pair $\{e_1, e_2\}$ from $\{\{e^*, p(e^*)\} : e^* \in E(G')\} \setminus (P_{uv} \cup Q_{uv})$ (the cardinality of this set is at least $k/2 - (8d + 16d^2/99) \geq d^2/6 - 16d^2/99 - 8d \geq 1$ for sufficiently large n) and change the pairing of the edges according to $p(e) := e_1$ and $p(e') := e_2$. Repeat this process until $|\{\{e^*, p(e^*)\} : e^* \in \hat{\Gamma}_{G'}(u) \wedge p(e^*) \in \hat{\Gamma}_{G'}(v)\}|$ drops below 100. □

Proof of Theorem 1.1 Let C be a sufficiently large absolute constant, such that for every sufficiently large positive integer n, and for every $t \in [dn/2 - 1, dn]$, where $d = \lfloor C \log n \rfloor$, it holds that $\lfloor C_1 \log t \rfloor \leq d - 1$ and $2d \leq t/30$, where C_1 is the constant from Lemma 2.1. In order to prove Theorem 1.1 it suffices to prove that for every sufficiently large n, if $G = (V, E)$ is a graph with n vertices and m edges, $\delta(G) \geq d$, and $D(G)$ is any orientation of G, then $D(G)$ admits an antimagic labeling.

It is convenient to split the description of the proof into five phases.

Phase 1: As long as there are two adjacent vertices in G, each having degree at least $d + 1$, assign the largest yet unused label to the edge that connects them and delete it. If the remaining graph contains an odd number of edges, pick some arbitrary edge, label it with the largest yet unused label and delete it. Let G' denote the spanning subgraph of G obtained at the end of this process. Each vertex $v \in V$ has a partial vertex sum, denoted by $r(v)$, which is the oriented sum of the labels of all edges from $\hat{\Gamma}(v)$ that were deleted during this phase (if no edge incident with v was labeled, then we set $r(v) = 0$). Let A denote the set of vertices of G' with degree $d - 1$ or d, and let $B = V \setminus A$. Note that the vertices of B form an independent set, and that $\deg_{G'}(v) \geq d + 1$ for every $v \in B$. Let $t \leq m$ denote the number of edges of G'. Note that t is even and that $t \in [dn/2 - 1, dn]$. Our goal is to assign the labels $\{1, \ldots, t\}$ to the edges of G' such that all vertex sums (cumulated with the partial sums $\{r(v) : v \in V\}$) will be distinct.

Phase 2: Apply Lemma 2.2 to the graph G', to obtain a function F and a pairing p that satisfy conditions (1)–(4) of the lemma.

Phase 3: Randomly partition the set $\{1, \ldots, t\}$ into $t/2$ pairwise disjoint pairs of labels. Assign those pairs of labels arbitrarily to the edge pairs determined by p. For every $e \in E(G')$, let $L(e)$ denote the pair of labels assigned to the edge pair $\{e, p(e)\}$.

Phase 4: For each $v \in B$, let $f(v)$ denote the oriented vertex sum of the labels assigned to the edges of $F(v)$. Note that although we have not yet specified which edge will get which label (for each edge there are two choices), $f(v)$ is well-defined by condition (2) from Lemma 2.2. For every $v \in A$, set $f(v) = 0$. For every $v \in B$, let $H(v) = \hat{\Gamma}_{G'}(v) \setminus F(v)$, and for every $v \in A$ let $H(v) = \hat{\Gamma}_{G'}(v)$. Note that $d-1 \le |H(v)| \le 2d$ (condition (1) from Lemma 2.2). For every vertex $v \in V$, there are $2^{|H(v)|}$ different ways of assigning the labels from $L(H(v))$ to the edges of $H(v)$, each assignment yielding a certain contribution $Q(v)$ to the oriented vertex sum of v (note condition (3) from Lemma 2.2 here). By Lemma 2.1, for each fixed vertex v with probability at least $1 - 1/t^2$, $Q(v)$ attains no integer s for more than a $C_2/(t|H(v)|^{1/2})$-fraction of the $2^{|H(v)|}$ assignments. It follows from a union bound argument that one can *fix* an assignment of label pairs to edge pairs, such that for *every* vertex v and for *every* integer s, it holds that $\Pr[Q(v) = s] \le \frac{C_2}{t|H(v)|^{1/2}} \le \frac{C_2}{t\sqrt{d-1}}$.

Phase 5: For every pair $\{e, p(e)\} \subseteq E(G')$ we flip a fair coin to decide which edge gets which label from $L(e)$; all $t/2$ coin flips are independent. Note that the final weight of each vertex v is given by $r(v) + f(v) + Q(v)$. We claim that with positive probability, no two vertices will end up with the same final weight. For every vertex v, let $J(v) \in \{1,2\}^{|H(v)|}$ denote the decision vector, showing the way in which the labels from $L(H(v))$ are assigned to every edge $e \in H(v)$ and its paired edge $p(e)$. For any two vertices $u, v \in V$, such that exactly l edges from $H(u) \cup p(H(u))$ have v as an endpoint, and for every decision vector $\vec{x} \in \{1,2\}^{|H(u)|}$, we have $\Pr[Q(v) = s \mid J(u) = \vec{x}] \le 2^l \Pr[Q(v) = s]$ for every integer s. By our choice of the pairing function p (see condition (4) from Lemma 2.2) we have $l \le 101$ (if u and v are not adjacent, then $l \le 100$), and thus the right hand side of this inequality is bounded from above by $2^{101}\frac{C_2}{t\sqrt{d-1}} \le 2^{101}\frac{C_2}{(dn/2-1)\sqrt{d-1}}$. For every two vertices $u, v \in V$, let $B(u,v)$ denote the event that both u and v end up with the same final weight. Denoting $s(u,v) := r(u) + f(u) - r(v) - f(v)$ we have

$$\begin{aligned}
\Pr[B(u,v)] &= \Pr[Q(v) = Q(u) + s(u,v)] \\
&= \sum_{\vec{x} \in \{1,2\}^{|H(u)|}} \Pr[J(u) = \vec{x}] \cdot \Pr[Q(v) = Q(u) + s(u,v) \mid J(u) = \vec{x}] \\
&\le 2^{101} \frac{C_2}{(dn/2-1)\sqrt{d-1}}.
\end{aligned}$$

We will prove that $B(u,v)$ is independent of all other events $B(x,y)$ $(x, y \in V)$ but at most $O(nd)$. Let Z denote the set of vertices that are endpoints of edges of $H(u) \cup H(v) \cup p(H(u)) \cup p(H(v))$. Clearly, $|Z| \le 6 \cdot 2d + 2$. Any event $B(x,y)$, where neither x nor y is in Z, is independent of $B(u,v)$. Thus $B(u,v)$ is independent of all but at most $(12d+2)n$ other events. Since

$$2^{101} \frac{C_2}{(dn/2-1)\sqrt{d-1}} \cdot (12d+2)n \le \frac{1}{3}$$

holds for sufficiently large n, we can apply the Lovász Local Lemma and conclude that with positive probability, no two oriented vertex sums are the same. It follows that $D(G)$ admits an antimagic labeling.

□

Proof of Theorem 1.2

1. The claim holds trivially for $n = 1$, so we can assume that $n \geq 3$. Let $D(S_n)$ be an arbitrary orientation of S_n. Let u denote the center of the star (the unique vertex of degree n). Let $I = \{x_1, \ldots, x_r\}$ denote the set of in-neighbors of u (that is, $(x_i, u) \in D(S_n)$ for every $1 \leq i \leq r$) and let $O = \{x_{r+1}, \ldots, x_n\}$ denote the set of out-neighbors of u. Assume without loss of generality that $r \geq n/2$ (reversing all edges of a graph results in multiplying every vertex-sum by -1; this preserves antimagic labelings). For every $1 \leq i \leq n$, assign the edge (u, x_i) the label $n + 1 - i$. Denote the resulting vertex sum of a vertex v by $\omega(v)$, then clearly

$$\omega(x_1) < \cdots < \omega(x_r) < \omega(x_n) < \cdots < \omega(x_{r+1}) < \omega(u),$$

where the last inequality follows since $r \geq n/2$ and $n \geq 3$.

2. For $3 \leq n \leq 10$, we have verified the claim by exhaustive enumeration, using a computer; hence we will assume that $n \geq 11$. Let $D(W_n)$ be an arbitrary orientation of W_n. Let u denote the center of the wheel. Let $I = \{x_1, \ldots, x_r\}$ denote the set of in-neighbors of u and let $O = \{x_{r+1}, \ldots, x_n\}$ denote the set of out-neighbors of u. Assume without loss of generality that $r \geq n/2$. We will also assume for now that $r \leq n - 3$. Assume for convenience that n is even (for odd n the proof is essentially the same). Assign the "cycle edges" (edges which are not incident with u) of $D(W_n)$ the labels $1, \ldots, n$ in the following clockwise order:

$$1, n, 2, n-1, 3, n-2, \ldots, n/2, n/2+1.$$

Denote the resulting vertex sum of a vertex v by $\omega(v)$; clearly, $-(n+2) \leq \omega(x_i) \leq n+2$ for every $1 \leq i \leq n$. Assume without loss of generality that $\omega(x_1) \leq \ldots \leq \omega(x_r)$ and that $\omega(x_{r+1}) \geq \ldots \geq \omega(x_n)$. Assign the edge (u, x_i) the label $2n + 1 - i$ and denote the resulting vertex sum of a vertex v by $\hat{\omega}(v)$. It is easy to see that the following conditions hold:

   (a) $\hat{\omega}(x_1) < \ldots < \hat{\omega}(x_r)$ and $\hat{\omega}(x_{r+1}) > \ldots > \hat{\omega}(x_n)$;

   (b) $\hat{\omega}(x_i) \leq -2$ for every $1 \leq i \leq r$ (since $r \leq n - 3$);

   (c) $\hat{\omega}(x_i) \geq -1$ for every $r + 1 \leq i \leq n$;

   (d) $\hat{\omega}(x_i) \leq 2.5n + 2$ for every $1 \leq i \leq n$ (since $r \geq n/2$).

   In order to prove that all vertex sums are distinct, it is therefore sufficient to prove that $\hat{\omega}(u) \neq \hat{\omega}(x_i)$ for every $1 \leq i \leq n$; by (d) above it suffices to prove that $\hat{\omega}(u) > 2.5n + 2$. We have

$$\hat{\omega}(u) = \sum_{i=1}^{r}(2n+1-i) - \sum_{j=r+1}^{n}(2n+1-j) = (2n+1)(2r-n) + \binom{n+1}{2} - 2\binom{r+1}{2}.$$

   It is clear that $\hat{\omega}(u)$ is minimized for $r = n/2$, entailing $\hat{\omega}(u) \geq n^2/4 > 2.5n + 2$, where the last inequality holds for $n \geq 11$.

The cases $r = n$ and $r = n-1$ are easy. The case $r = n-2$ is slightly more involved, but still easy. The main idea is to label the cycle edges that are incident with x_{n-1} and with x_n with the labels $1, 3, 2$ if they are adjacent, and with the labels $1, 4$ and $2, 3$ respectively, otherwise. The rest of the cycle edges will be labeled similarly to the proof above, entailing $-5 \leq \omega(x_{n-1}), \omega(x_n) \leq 5$, and $-(n+6) \leq \omega(x_i) \leq n+6$ for every $1 \leq i \leq n-2$. This is enough to ensure $\hat{\omega}(x_1) < \cdots < \hat{\omega}(x_{n-2}) < \hat{\omega}(x_n) < \hat{\omega}(x_{n-1}) < \hat{\omega}(u)$.

3. Before proving that any orientation of K_n ($n \neq 3$) is antimagic, we prove a lemma that might be of independent interest. This lemma determines precisely the values $n, m \in \mathbb{N}$ for which there exists a simple graph G with n vertices and m edges, such that the degree of every vertex of G is even (we will call such graphs *even degree graphs*). It turns out that such graphs exist for almost all pairs (n, m) with $0 \leq m \leq \binom{n}{2}$.

Lemma 2.3 *Let $n \in \mathbb{N}$ and let $m \in \{0, 1, \ldots, \binom{n}{2}\}$. There exists a simple even degree graph with n vertices and m edges, if and only if $m \notin \{1, 2, \binom{n}{2} - 2, \binom{n}{2} - 1\}$ for odd n, and $m \notin \{1, 2, \binom{n}{2} - \frac{n}{2} + 1, \binom{n}{2} - \frac{n}{2} + 2, \ldots, \binom{n}{2}\}$ for even n.*

Proof Clearly, no single edge and no pair of edges can comprise an even degree graph. On the other hand, if n is odd, then $G \subseteq K_n$ is an even degree graph iff $K_n \setminus G$ is an even degree graph; this excludes the values $m \in \{\binom{n}{2} - 2, \binom{n}{2} - 1\}$. If n is even, then all vertices of K_n have odd degree and at least $n/2$ edges (a perfect matching) have to be removed from K_n to give an even degree graph; this excludes the values $\binom{n}{2} - \frac{n}{2} + 1, \binom{n}{2} - \frac{n}{2} + 2, \ldots, \binom{n}{2}$.

In order to prove that there exists an even degree graph for all of the remaining values of m, we proceed by induction on n. The claim is easily verified for all $n \leq 5$.

Let $n > 5$ and assume first that n is even. By the induction hypothesis there exists an even degree graph on $n-1$ vertices with exactly m edges for every $m \in \{0, 3, 4, \ldots, \binom{n-1}{2} - 4, \binom{n-1}{2} - 3, \binom{n-1}{2}\}$. Adding an isolated vertex to one of these graphs yields an even degree graph on n vertices; this proves our claim for these values of m. Let G be an even degree graph on $n-1$ vertices with exactly $\binom{n-1}{2} - 3$ edges. Remove an arbitrary edge (x, y) from G, and add a new vertex u and the two edges $(u, x), (u, y)$ to obtain an even degree graph on n vertices with exactly $\binom{n-1}{2} - 2$ edges. Starting with K_n, remove a perfect matching $(u_1, v_1), \ldots, (u_{n/2}, v_{n/2})$ to obtain an even degree graph H on n vertices with exactly $\binom{n}{2} - n/2$ edges. Let $1 \leq k \leq n/2$. For every $1 \leq i \leq k$, replace the pair of edges (u_i, u_{i+1}), (v_i, u_{i+1}) of H with the single edge (u_i, v_i) to obtain an even degree subgraph of K_n with exactly $\binom{n}{2} - n/2 - k$ edges. The case of even n now follows, as $\{0, 3, 4, \ldots, \binom{n-1}{2} - 4, \binom{n-1}{2} - 3, \binom{n-1}{2}\} \cup \{\binom{n-1}{2} - 2\} \cup \{\binom{n}{2} - n, \ldots, \binom{n}{2} - n/2\} = \{0, 3, 4 \ldots, \binom{n}{2} - n/2\}$.

Assume now that $n > 5$ is odd. As in the previous case, an application of the induction hypothesis and the addition of an isolated vertex yields even degree graphs

G_m on n vertices with exactly $m \in \{0, 3, 4, \ldots, \binom{n-1}{2} - \frac{n-1}{2}\}$ edges. Since n is odd, $K_n \setminus G_m$ is also an even degree graph; clearly, it has exactly $\binom{n}{2} - m$ edges. The case of odd n now follows as well since $\{0, 3, 4, \ldots, \binom{n-1}{2} - \frac{n-1}{2}\} \cup \{\binom{n}{2} - \binom{n-1}{2} + \frac{n-1}{2}, \ldots, \binom{n}{2} - 4, \binom{n}{2} - 3, \binom{n}{2}\} = \{0, 3, 4, \ldots, \binom{n}{2} - 4, \binom{n}{2} - 3, \binom{n}{2}\}$ holds for $n > 5$. □

We are now ready to prove our claim. For $n \in \{1, 2, 4, 5\}$ the claim can be easily verified; thus we can assume that $n \geq 6$. Let $D(K_n)$ be an arbitrary orientation of K_n. Let u be a vertex of $D(K_n)$ with maximum in-degree, let $I = \{x_1, \ldots, x_r\}$ denote the set of in-neighbors of u and let $O = \{x_{r+1}, \ldots, x_{n-1}\}$ denote the set of out-neighbors of u. Note that $r \geq \lceil \frac{n-1}{2} \rceil$ by our assumption of maximality. Let L_I denote the set of the largest r labels of the same parity as $\binom{n}{2}$, and let L_O denote the set of the smallest $n - 1 - r$ labels of the opposite parity. Assign the labels of $\{1, 2, \ldots, \binom{n}{2}\} \setminus (L_I \cup L_O)$ to the edges of $D(K_n) \setminus \{u\}$, such that the edges which are assigned odd labels span an even degree subgraph (such an even degree subgraph exists due to Lemma 2.3 which applies since $n \geq 6$ and thus the number of odd labels is ≥ 2 and clearly small enough). Denote the resulting vertex sum of a vertex v by $\omega(v)$. Assume without loss of generality that $\omega(x_1) \leq \cdots \leq \omega(x_r)$ and that $\omega(x_{r+1}) \leq \cdots \leq \omega(x_{n-1})$. By this labeling, $\omega(v)$ is even for every vertex $v \in V(K_n)$. Assign the labels from L_I in descending order to the edges $(u, x_1), \ldots, (u, x_r)$ and the labels from L_O in ascending order to the edges $(u, x_{r+1}), \ldots, (u, x_{n-1})$. Denote the resulting vertex sum of a vertex v by $\hat{\omega}(v)$. The following conditions hold:

(a) $\hat{\omega}(x_1) < \cdots < \hat{\omega}(x_r)$ and $\hat{\omega}(x_{r+1}) < \cdots < \hat{\omega}(x_{n-1})$;

(b) For every $1 \leq i \leq r$ and every $r+1 \leq j \leq n-1$ we have $\hat{\omega}(x_i) \not\equiv \hat{\omega}(x_j) \pmod 2$; in particular $\hat{\omega}(x_i) \neq \hat{\omega}(x_j)$;

In order to prove that all vertex sums are distinct, it is therefore sufficient to prove that $\hat{\omega}(u) > \hat{\omega}(x_i)$ for every $1 \leq i \leq n - 1$. Recall that u is a vertex with maximum in-degree r; thus, by our labeling $\hat{\omega}(u) \geq \hat{\omega}(x_i)$ for every $1 \leq i \leq n - 1$. Assume for the sake of contradiction that there exists some $1 \leq i \leq n-1$ for which $\hat{\omega}(u) = \hat{\omega}(x_i)$. It follows that u and x_i have the same in-degree $r = \frac{n-1}{2}$, that

$$\hat{\omega}(u) = \sum_{i=0}^{r-1} \left(\binom{n}{2} - 2i \right) - \sum_{i=1}^{r} 2i,$$

and that

$$\hat{\omega}(x_i) = \sum_{i=0}^{r-1} \left(\binom{n}{2} - 2i - 1 \right) - \sum_{i=1}^{r} (2i - 1).$$

This is clearly impossible, because the label of the edge (u, x_i) has to appear in both sums.

□

3 Graphs which admit an antimagic orientation

The main idea of the proofs of Theorems 1.3 and 1.4 is to use eulerian orientations. To start with the simplest case, given a eulerian graph, one can orient all edges along a eulerian cycle, and label them in consecutive increasing order along this cycle. If the graph is regular this gives a labeling where all vertex sums except one (the start and end vertex of the cycle) are the same. We then perturb this almost magic labeling and obtain an antimagic labeling by flipping the orientation of certain arcs or removing them. Depending on which type of graph we start with, different methods on how to make it eulerian and how to perturb an almost magic labeling are used.

Proof of Theorem 1.3

Let $G = (V, E)$ be a $(2d + 1)$-regular graph with $2n$ vertices and m edges. Let $M = \{(u_i, v_i) : 1 \leq i \leq n\}$ be a perfect matching of the vertices of V such that $H := G \cup M$ is a connected multigraph (it might contain parallel edges); such a matching M exists as every connected component of G is of size at least 2. Let $D(H)$ be an arbitrary eulerian orientation of H; assume without loss of generality that (u_i, v_i) is directed from u_i to v_i in $D(H)$ for every $1 \leq i \leq n$. We consider the eulerian cycle of $D(H)$ as if it ends with the edge (u_1, v_1); clearly, this means that the cycle starts with some edge $(v_1, w) \in E$ which is directed from v_1 to w in $D(H)$. Label the edges of H according to the ordering dictated by this eulerian orientation as follows. The edge (v_1, w) will get the label 1. For every other edge, if it is an edge of M, it will get the same label as the previous edge, whereas if it is an edge of G, it will get the successor of that label. Denote this edge labeling by f. It is evident that f, when restricted to the edges of G, is a bijection from E to the integers $\{1, \ldots, m\}$ and that the restriction of f to the edges of M is injective. Moreover, the current vertex sum of v_1 is $m-(d+1)$, the current vertex sum of v_i is $-(d+1)$ for every $2 \leq i \leq n$, and the current vertex sum of u_i is $-d$ for every $1 \leq i \leq n$. Remove the edges of M and denote the resulting vertex sum of $x \in V$ by $\omega(x)$. For $1 \leq i \leq n$ let a_i denote the label that was assigned to the edge (u_i, v_i). It is easy to see that $\omega(v_1) = -(d + 1)$, $\omega(v_i) = -(d + 1) - a_i < \omega(v_1)$ for every $2 \leq i \leq n$ and $\omega(u_i) = -d + a_i > \omega(v_1)$ for every $1 \leq i \leq n$. It follows that the restriction of f to the edges of G is an antimagic labeling of $D(G)$.

□

Proof of Theorem 1.4

Let $G = (V, E)$ be a $2d$-regular graph, and let $M = \{(u_i, v_i) : 1 \leq i \leq n\}$ be a matching that covers all vertices of G, but at most one. Note that the number of vertices of G is either $2n$ or $2n + 1$.

First consider the case that M is perfect. Let $D(G)$ be an arbitrary eulerian orientation of G. Starting with the edge (u_1, v_1), label the edges of G with the numbers $1, \ldots, 2dn$ in increasing order along a eulerian cycle of $D(G)$. The current vertex sum of u_1 is $d(2n-1)$, whereas the current vertex sum of any other vertex is $-d$. Now switch the orientation of all edges of M. Observe that switching the orientation of an edge with the label a changes the vertex sums of its endpoints by $+2a$ and $-2a$, respectively. For every vertex $v \in V$, let $\omega(v)$ denote the new vertex sum of v. As the labels of the matching edges differ by at least 2, for every $u, v \in V \setminus \{u_1\}$ we have $|\omega(u) - \omega(v)| \geq 4$; in particular, all of these vertex sums are distinct. Hence there is only one potential conflict, a conflict between the vertex sums at u_1 and at some $v \in V \setminus \{u_1\}$. In this case, switch back the orientation of the edge (u_1, v_1). This changes the vertex sums of u_1 and v_1 by -2 and $+2$ back to $d(2n-1)$ and $-d$ and thus resolves all conflicts.

Now suppose one vertex s remains unmatched by M. Let $D(G)$ be a eulerian orientation of G, that admits a eulerian cycle C, in which the edges (s, u_1) and (u_1, v_1) appear consecutively (it is not hard to see that such a eulerian orientation exists; possibly one will have to rename matching edges). Starting with the edges (s, u_1) and (u_1, v_1), label the edges of G with the numbers $1, \ldots, d(2n+1)$ in increasing order along the eulerian cycle C of $D(G)$. The current vertex sum of s is $2dn$, whereas the current vertex sum of any other vertex is $-d$. Now switch the orientation of all edges of M. For every vertex $v \in V$, let $\omega(v)$ denote the new vertex sum of v. Note that the edge (u_1, v_1) is assigned the label 2, and thus we have $\omega(u_1) = -d + 4$. For every $u, v \in V \setminus \{s\}$ we have $|\omega(u) - \omega(v)| \geq 4$; in particular, all of these vertex sums are distinct. Hence there is only one potential conflict, a conflict between the vertex sums at s and at some $v \in V \setminus \{s\}$. In this case switch the orientation of the edge (s, u_1). This changes the vertex sums of s and u_1 by $+2$ and -2 to $2dn + 2$ and $-d + 2$ and thus resolves all conflicts.

□

Proof of Theorem 1.6

In the proof of Theorem 1.6 we will use the following lemma from [7].

Lemma 3.1 *Let r be a positive integer and let $r = d_1 + \cdots + d_n$ be a partition of r with $d_i \geq 2$ for every $1 \leq i \leq n$. Define $r' := r + 1$ if r is even, and $r' := r$ if r is odd. Then the set $\{1, \ldots, r\}$ can be partitioned into pairwise disjoint sets $X_1, \ldots, X_n$, such that for every $1 \leq i \leq n$ we have $|X_i| = d_i$ and $\sum_{x \in X_i} x \equiv 0 \pmod{r'}$.*

We will also use the following lemma, whose proof appears after the proof of the theorem.

Lemma 3.2 *Let $a \in \mathbb{N}$. Any graph $G = (V, E)$ with m edges admits an orientation and an edge-labeling with the labels $\{a + 1, a + 2, \ldots, a + m\}$, such that every oriented vertex sum is at most $a + m$.*

We are now ready to prove Theorem 1.6. Let m denote the number of edges of G, let M be a perfect matching of G, and let $u_1, \dots, u_n$ be the vertices of the independent set U, and let $W = V \setminus U$ (note that $\Gamma(U) = W$). Set $d_i := \deg(u_i) - 1$ for every $1 \le i \le n$, and $r := d_1 + \cdots + d_n$; note that $d_i \ge 2$ for every $1 \le i \le n$. We will label the non-matching edges that are incident with U using the smallest possible labels $\{1, \dots, r\}$. The largest n labels $\{m - n + 1, \dots, m\}$ will be reserved for the edges of M, and the remaining labels $\{r + 1, \dots, m - n\}$ for the edges of $G' := G[W]$.

By Lemma 3.2 one can orient and label the edges of G', using the labels $\{r+1, \dots, m-n\}$, such that the vertex sum at every vertex $v \in V(G')$ is at most $m - n$.

Next, orient all edges that are incident with U (including the edges of M) towards U. By Lemma 3.1 there is a partition of the set $\{1, \dots, r\}$ into pairwise disjoint sets $X_1, \dots, X_n$ such that for all $1 \le i \le n$ we have $|X_i| = d_i$ and $\sum_{x \in X_i} x \equiv 0 \pmod{r'}$ where r' is defined as in the lemma. Use the labels of X_i for the non-matching edges incident with u_i arbitrarily.

Denote the resulting vertex sum of a vertex v by $\omega(v)$. The function ω defines an ordering $v_1, \dots, v_n$ of the vertices of W such that, without loss of generality, $\omega(v_1) \ge \cdots \ge \omega(v_n)$. For every $1 \le i \le n$ assign the label $m - n + i$ to the matching edge which is incident with v_i. Denote the resulting vertex sum of a vertex v by $\hat{\omega}(v)$. Note that the following conditions hold:

1. $\hat{\omega}(v_i) \le \underbrace{\omega_{G'}(v_i)}_{\le m-n} - (m - n + 1) < 0$ ($\omega_{G'}(v_i)$ denotes the partial oriented vertex sum of v_i, obtained by considering only the labels given to edges of G'), and $\hat{\omega}(u_i) > 0$ for every $1 \le i \le n$;
2. $\hat{\omega}(v_1) > \cdots > \hat{\omega}(v_n)$;
3. As $n < r \le r'$ we have $m - n + i \not\equiv m - n + j \pmod{r'}$ for all $1 \le i < j \le n$. Since $\omega(u_i) \equiv \omega(u_j) \equiv 0 \pmod{r'}$, it follows that $\hat{\omega}(u_i) \not\equiv \hat{\omega}(u_j) \pmod{r'}$, and therefore $\hat{\omega}(u_i) \ne \hat{\omega}(u_j)$ for all $1 \le i < j \le n$.

It follows that all vertex sums are distinct.

□

Proof of Lemma 3.2

If $m = 0$, then there is nothing to prove; hence we may assume that $m \ge 1$. First, add a set E' of auxiliary edges to G to obtain a (multi)graph $G' = (V, E \cup E')$ such that G' is eulerian (that is, G' is connected, and every vertex of G' has an even degree), every vertex $v \in V$ is incident with at most two edges of E', and there exists a vertex $s \in V$ that is incident with at most one edge of E'. It is easy to see that one may choose a eulerian orientation D of G' such that, if two edges from E' are incident with a vertex v, then exactly one of them is directed towards v and if s is incident with one of the edges of E', then this edge is directed towards s. This implies that in the subgraph H of D, spanned by the edges of E', every vertex $v \in V$ has out-degree at most one.

Now we label the edges of G along the eulerian cycle of $D(G')$, using the labels $\{a+1, a+2, \ldots, a+m\}$ in increasing order for the edges of G, and reusing the label of the previous edge whenever we encounter an edge of E'. We start this labeling at the vertex s, such that, if there is an edge of E' incident with s, then it is the last edge of the cycle (if $\deg_H(s) = 0$, then just start with any edge incident with s).

Denote the resulting vertex sum of every vertex v of G' by $\omega'(v)$. Denote by $\omega(v)$ the partial oriented vertex sum of v, obtained by considering only the labels given to edges of G.

As the labels along the eulerian cycle of $D(G')$ are monotone increasing, we have $\omega'(v) \leq 0$ for every $v \in V \setminus \{s\}$ and $\omega'(s) = -(\deg_{G'}(s)/2 - 1) - (a+1) + (a+m) \leq a+m$. As the out-degree in H of every vertex $v \in V$ is at most one, we have $\omega(v) \leq \omega'(v) + (a+m) \leq a+m$ for every $v \in V \setminus \{s\}$. Moreover, $\omega(s) \leq \omega'(s) \leq a+m$.

□

Remark The bound $a+m$ in Lemma 3.2 is tight. Consider for instance a graph consisting only of a single edge (or more generally, a matching), then, essentially, there is only one possible orientation and labeling, and one of the vertices will receive the vertex sum $a+m$.

4 Concluding remarks and open problems

Antimagic orientations: It follows immediately from Theorem 1.1 that every "dense" graph admits an antimagic orientation (in fact, many such orientations). Moreover, we have proved that many "sparse" graphs (including almost all regular graphs) admit an antimagic orientation. This leads us to make the following conjecture:

Conjecture 4.1 *Every connected undirected graph admits an antimagic orientation.*

All orientations antimagic: The only connected directed graphs, that we know are not antimagic are $K_{1,2}$ and K_3, directed such that the out-degree and the in-degree of every vertex are at most 1. This motivates us to ask the following question:

Question 4.2 *Is it true that every connected directed graph with at least* 4 *vertices is antimagic?*

By exhaustive enumeration using a computer, we have verified that every connected directed graph on $4 \leq n \leq 7$ vertices admits an antimagic labeling. This shows some support for an affirmative answer of Question 4.2.

Directed and undirected antimagicness: As was indicated in the Introduction, if a bipartite graph admits an antimagic labeling, then it also admits an antimagic orientation. If the assertion of Conjecture 4.1 is true, then this implication holds trivially for every connected graph. However, in general, this does not seem to be an easy claim to prove. Indeed, a natural way to prove such an implication would be to fix an antimagic labeling of the undirected graph, and then to somehow find an orientation which is antimagic with the same labeling. While this can be done with bipartite graphs, it is not possible in general. Consider for example an undirected graph G, consisting of $r \geq 5$ vertex disjoint triangles. For every $1 \leq i \leq r$, assign the labels $3i, 3i-1, 3i-2$ to the edges of the ith triangle arbitrarily. It is easy to see that this yields an antimagic labeling of G. Now, let D be any orientation of G. In every triangle of D there is a vertex of out-degree exactly one. Its oriented vertex sum must lie in $\{-2, -1, 1, 2\}$. Since $r \geq 5$, there will be two vertices of D with the same vertex sum. This example also shows, that while it is possible that every orientation of a graph admits an antimagic labeling (see Question 4.2 above), the converse is not true (at least if we allow unconnected graphs); that is, there exists an antimagic labeling f of an undirected graph G such that f is not an antimagic labeling of any orientation D of G.

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