Rainbow Hamilton cycles in randomly coloured randomly perturbed dense graphs

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November 29, 2020

Abstract

Given an n-vertex graph H with minimum degree at least dn for some fixed $d > 0$, the distribution $H \cup \mathbb{G}(n,p)$ over the supergraphs of H is referred to as a (random) *perturbation* of H. We consider the distribution of edge-coloured graphs arising from assigning each edge of the random perturbation $H \cup \mathbb{G}(n,p)$ a colour, chosen independently and uniformly at random from a set of colours of size $r := r(n)$. We prove that edge-coloured graphs which are generated in this manner a.a.s. admit rainbow Hamilton cycles whenever the edge-density of the random perturbation satisfies $p := p(n) \geq C/n$, for some fixed $C > 0$, and $r = (1 + o(1))n$. The number of colours used is clearly asymptotically best possible. In particular, this improves upon a recent result of Anastos and Frieze (2019) in this regard. As an intermediate result, which may be of independent interest, we prove that randomly edge-coloured sparse pseudo-random graphs a.a.s. admit an almost spanning rainbow path.

1 Introduction

A classical result of Dirac [11] asserts that every n-vertex graph H (on at least three vertices) with minimum degree $\delta(H) \geq n/2$ is Hamiltonian. Moreover, Dirac's result is optimal as far as the constant $1/2$ appearing in the condition on $\delta(H)$ is concerned.

Let $\mathcal{G}_{d,n}$ denote the set of n-vertex graphs with minimum degree at least dn for some constant $d > 0$. As noted above, for every $d \in (0, 1/2)$, there are non-Hamiltonian graphs $H \in \mathcal{G}_{d,n}$. Nevertheless, Bohman, Frieze, and Martin [7] discovered that once *slightly* randomly perturbed (i.e. smoothed), the members of $\mathcal{G}_{d,n}$ almost surely give rise to Hamiltonian graphs. In particular, they proved that for every $d > 0$ there exists a constant $C := C(d)$ such that $H \cup \mathbb{G}(n,p)$ is asymptotically almost surely (a.a.s. for brevity, hereafter) Hamiltonian whenever $H \in \mathcal{G}_{d,n}$ and $p := p(n) \geq C/n$,

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undershooting the threshold for Hamiltonicity in the binomial random graph model $\mathbb{G}(n,p)$ by a logarithmic factor. Numerous results regarding spanning configurations in members of the graph distribution $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p)$ (and its hypergraph analogue) have since appeared; see, e.g., [4, 5, 6, 8, 9, 12, 16, 17, 19, 20, 22].

Fix an arbitrary graph $H \in \mathcal{G}_{d,n}$. Let $\mathscr{G}(H,p,r)$ be the distribution of edge-coloured supergraphs of H which is defined as follows. Perturb H using $\mathbb{G}(n,p)$, and colour the edges of the perturbation $H \cup \mathbb{G}(n,p)$ by assigning each edge a colour chosen independently and uniformly at random from the set $[r]$.

Anastos and Frieze [2] proved that for every $d > 0$ there exists a constant $C := C(d)$ such that if $p := p(n) \geq C/n$, $r > (120 - 20 \ln d)n$ and H is any graph in $\mathcal{G}_{d,n}$, then a.a.s. $\mathscr{G}(H,p,r)$ admits a *rainbow*[1] Hamilton cycle. The corresponding problem for random graphs was extensively studied. In particular, improving upon earlier results, Frieze and Loh [15] proved that randomly colouring the edges of $\mathbb{G}(n, (1+o(1)) \ln n/n)$ with $(1+o(1))n$ colours a.a.s. yields a rainbow Hamilton cycle. Both the edge-density of the random graph and the number of colours asserted by their result are clearly asymptotically best possible; nevertheless, both were refined. Ferber and Krivelevich [14] improved the result of Frieze and Loh by replacing $p = (1+o(1)) \ln n/n$ with the optimal $p = (\ln n + \ln \ln n + \omega(1))/n$. Bal and Frieze [3] proved that if $p = \omega(\ln n/n)$, then precisely n colours suffice. Ferber [13] improved the latter result to $p = \Omega(\ln n/n)$.

Our main result asserts that in the perturbed setting with random perturbations of edge-density C/n, a set of colours of size $(1+o(1))n$ suffices in order to yield a rainbow Hamilton cycle asymptotically almost surely. This improves upon the aforementioned result of Anastos and Frieze [2] in terms of the number of colours used, which is clearly asymptotically best possible.

Theorem 1.1. *For every $d, \alpha > 0$ there exists a constant $C := C(d,\alpha)$ such that for every $H \in \mathcal{G}_{d,n}$, $\mathscr{G}(H,p,r)$ a.a.s. contains a rainbow Hamilton cycle, whenever $p := p(n) \geq C/n$ and $r = (1+\alpha)n$.*

Remark 1.2. Our proof of Theorem 1.1 can be adapted in a straightforward manner to yield its directed analogue. That is, one can prove that adding $\Theta(n)$ random directed edges to a digraph on n vertices with minimum in-degree and minimum out-degree dn, where $d > 0$ is an arbitrarily small constant, and then randomly colouring the edges of the augmented digraph with $(1+o(1))n$ colours, a.a.s. gives rise to a directed rainbow Hamilton cycle.

Remark 1.3. It would be interesting to know whether Theorem 1.1 can be extended to handle $(1+\alpha)n$ colours, for $\alpha := \alpha(n) = o(1)$ (in a meaningful manner) while C remains fixed. Our current proof does not allow this. In particular, it would be interesting to know the smallest value of $C = C(n)$ for which the result holds with $\alpha = 0$. The aforementioned result of Ferber [13] implies that $C = \Omega(\ln n)$ suffices, but this bound is unlikely to be best possible for the perturbed model.

[1] A Hamilton cycle whose edges are coloured using n distinct colours.

In order to prove Theorem 1.1 we expose the edges of $\mathbb{G}(n,p)$ in two stages. First, we generate $R_1 \sim \mathbb{G}(n, c_1/n)$ and colour its edges uniformly at random from the colour set $[(1+\alpha)n]$. Using a rainbow variant of the DFS algorithm we show that a.a.s. R_1 admits an almost spanning rainbow path P; this is done in Section 2. Next, we use the edges of H to extend P to a rainbow path P' on $n-2$ vertices. This is done by absorbing vertices one by one. Whenever we wish to absorb a vertex, we expose the colours of the relevant edges of H. The absorbing machinery and the way we use it are discussed in Section 3. Finally, we use the remaining two vertices and the edges of $R_2 \sim \mathbb{G}(n, c_2/n)$ in order to close P' to a rainbow Hamilton cycle; this is discussed in Section 3 as well.

2 Almost spanning rainbow paths in sparse pseudorandom graphs

2.1 Rainbow DFS

In this section, we put forth an adaptation of the well-known DFS algorithm, to which we refer as *rainbow DFS* (RDFS for brevity, hereafter), fit for edge-coloured graphs. We then employ RDFS in order to produce "long" rainbow paths. In particular, the main result of this section is Proposition 2.1, stated below, which can be viewed as a rainbow version of [18, Proposition 2.2].

RDFS algorithm. The input for the RDFS algorithm consists of a graph G with vertex-set $[n]$, an edge-colouring $\psi : E(G) \to \mathbb{N}$ of G, and a permutation $\pi \in S_n$. During its execution, the algorithm maintains three sets of vertices, namely S, T and U, as well as a set of colours denoted by A_U. The set S consists of all vertices of G whose exploration is complete; the set T consists of all vertices of G that were not yet visited; finally, $U := [n] \setminus (S \cup T)$. The members of U are kept in a stack. Given $U = \{u_1, u_2, \ldots, u_t\}$, we maintain the convention that for every $1 \le i < j \le t$ the vertex u_i is pushed into U prior to u_j. The set of colours A_U is given by

$$A_U = \{\psi(u_i u_{i+1}) : 1 \le i \le t-1\}.$$

Initially $S = U = A_U = \emptyset$ and $T = [n]$; RDFS proceeds in rounds until $T = U = \emptyset$ and $S = [n]$. In rounds for which $U = \emptyset$, RDFS chooses the first vertex in T (according to π), deletes it from T, and pushes it into U. In each round for which $U \neq \emptyset$, the following actions are performed. Let $U = \{u_1, u_2, \ldots, u_t\}$ and $T = \{w_1, \ldots, w_m\}$, where the elements of T are ordered according to π (that is, $\pi(w_i) < \pi(w_j)$ if and only if $i < j$). If there exists an $i \in [m]$ such that $u_t w_i \in E(G)$ and $\psi(u_t w_i) \notin A_U$, then for the smallest such i the vertex w_i is deleted from T and pushed into U. If no such i is found, then u_t is popped out of U and added to S.

The following properties are maintained by RDFS.

(D1) In every round of the algorithm, either one vertex is moved from T to U or one vertex is moved from U to S.

(D2) $|\{\psi(e) : e \in E_G(S,T)\}| \le n-1$ holds at any point during the execution of the algorithm.

(D3) If $U = \{u_1, \ldots, u_t\}$, then $u_1 u_2 \ldots u_t$ is a path in G which is rainbow under ψ (in particular, A_U is well-defined).

While Properties (D1) and (D3) are immediate, Property (D2) merits a brief explanation. Consider an arbitrary $s \in S$. At the point in time where s is moved from U into S, every edge connecting s to T must have its colour present in A_U (where here S, T, U and A_U are the "snapshots" of these sets corresponding to the moment in time under examination). Observe that T does not increase over time and thus neither does $\{\psi(st) : t \in T\}$. Hence, the set of colours $\{\psi(e) : e \in E_G(S,T)\}$ is a subset of the union of the sets A_U taken over all rounds of the execution of the algorithm. Since (just like in the usual DFS algorithm) all sets U span paths whose union over all rounds of the execution of the algorithm is a forest, the aforementioned union of the sets A_U is of size at most $n-1$.

We are ready to state and prove the main result of this section.

Proposition 2.1. *Let $k < n$ be positive integers and let B be a set of at least n colours. Let G be a graph with vertex-set $[n]$ and let $\psi : E(G) \to B$. If $|\{\psi(e) : e \in E_G(X,Y)\}| \geq n$ holds for every pair of disjoint sets $X, Y \subseteq V(G)$ of size $|X| = |Y| = k$, then G admits a path of length $n - 2k$ which is rainbow under ψ.*

Proof. Run RDFS over G, ψ, and an arbitrary permutation $\pi \in S_n$. Consider the moment in time during the execution of the algorithm at which $|S| = |T|$; such a moment must exist by Property (D1). Since $|\{\psi(e) : e \in E_G(S,T)\}| \leq n-1$ holds by Property (D2), it follows, by assumption, that $|S| = |T| \leq k-1$, implying that $|U| \geq n - 2k + 2$. The proof is now complete since U spans a path in G which is rainbow under ψ by Property (D3). □

2.2 Long rainbow paths in jumbled graphs

An n-vertex graph G is said to be (p,β)-*jumbled* if

$$\left| e_G(X,Y) - p|X||Y| \right| \leq \beta \sqrt{|X||Y|}$$

holds for every $X, Y \subseteq V(G)$. The canonical examples of such graphs are the so-called (n,d,λ)-*graphs* (see, e.g., [21]) and random graphs. The latter, for instance, satisfy $\beta = \Theta(\sqrt{pn})$ asymptotically almost surely. More generally, $\beta \geq \sqrt{pn}$ is compelled [21]. Below we assume that $\beta \leq pn/D$, for some constant D, which in turn imposes a lower bound on p. Indeed, $\sqrt{pn} \leq \beta \leq pn/D$ implies that $p \geq D^2/n$. The reason we require such a lower bound on p will become apparent towards the end of this section.

The following result asserts that randomly edge-coloured pseudorandom graphs, satisfying a fairly mild discrepancy condition, a.a.s. admit almost-spanning rainbow paths.

Theorem 2.2. *For every $\alpha > 0$ and $\varepsilon > 0$ there exists a constant $D = D(\alpha, \varepsilon)$ such that the following holds whenever $\beta := \beta(n) \leq pn/D$ and n is sufficiently large. Let G be an n-vertex (p,β)-jumbled* *graph and let ψ be an edge-colouring of G assigning every edge of G a colour from $[(1+\alpha)n]$, chosen independently and uniformly at random. Then a.a.s. G admits a path of length $(1-\varepsilon)n$ which is rainbow under ψ.*

Proof. Given α and ε, set $D \gg \alpha^{-1}, \varepsilon^{-1}$. By Proposition 2.1 with $k = \varepsilon n/2$ (which we assume is an integer) and the set of colours $B = [(1+\alpha)n]$, it suffices to show that a.a.s. $|\{\psi(e) : e \in E_G(X,Y)\}| \geq n$ holds for every pair of disjoint sets $X, Y \subseteq V(G)$ of size $|X| = |Y| = k$. In order to do so, we prove that for every subset $A \subseteq [(1+\alpha)n]$ of size αn (which we assume is an integer) and every pair of disjoint sets $X, Y \subseteq V(G)$ of size $|X| = |Y| = k$ it holds that $\{\psi(e) : e \in E_G(X,Y)\} \cap A \neq \emptyset$.

By the jumbledness condition put on G,

$$e_G(X,Y) \geq p|X||Y| - \beta\sqrt{|X||Y|} = pk^2 - \beta k$$

holds for every pair of disjoint set $X, Y \subseteq V(G)$ of size k each. In particular, for $k = \varepsilon n/2$, we have that

$$pk^2 - \beta k \geq p\varepsilon^2 n^2/4 - p\varepsilon n^2/(2D) \geq p\varepsilon^2 n^2/8,$$

where the last inequality holds by our assumption that $D \gg \varepsilon^{-1}$. Then, the probability that $\{\psi(e) : e \in E_G(X,Y)\} \cap A = \emptyset$ holds for any such pair X, Y and set of colours A (of size αn) is bounded from above by

$$\begin{aligned}\binom{(1+\alpha)n}{\alpha n}\binom{n}{k}^2\left(1 - \frac{\alpha n}{(1+\alpha)n}\right)^{p\varepsilon^2 n^2/8} &\leq \left(\frac{e(1+\alpha)}{\alpha}\right)^{\alpha n}\left(\frac{2e}{\varepsilon}\right)^{\varepsilon n} e^{-\frac{\alpha}{8(1+\alpha)}\cdot p\varepsilon^2 n^2} \\ &\leq \exp\left\{2\ln(\alpha^{-1})\alpha n + 2\ln(\varepsilon^{-1})\varepsilon n - \alpha p\varepsilon^2 n^2/16\right\} \\ &= o(1).\end{aligned}$$

The last equality holds since (as noted in the paragraph preceding the statement of Theorem 2.2) the assumed upper bound on β implies that $p \geq D^2/n$, and D is sufficiently large with respect to α^{-1} and ε^{-1}. □

We conclude this section with the following direct consequence of Theorem 2.2 which is a rainbow version of a classical result of Ajtai, Komlós and Szemerédi [1] and independently of Fernandez de la Vega [10].

Corollary 2.3. *For positive constants α and ε there exists a constant $K = K(\alpha, \varepsilon)$ such that the following holds. Let $G \sim \mathbb{G}(n, K/n)$ and let ψ be a colouring assigning every edge of G a colour from $[(1+\alpha)n]$, chosen independently and uniformly at random. Then a.a.s. G admits a path of length $(1-\varepsilon)n$ which is rainbow under ψ.*

Remark 2.4. The fact that $\mathbb{G}(n, K/n)$ a.a.s. satisfies the discrepancy condition set in Theorem 2.2 (with constant β, depending solely on K) follows by a standard application of Chernoff's bound and a union bound. One can also prove Corollary 2.3 directly (i.e., without relying on Theorem 2.2), and such a proof avoids the use of Chernoff's bound.

3 Rainbow Hamilton cycles in the perturbed model

In this section, we prove Theorem 1.1. The main ingredients of our proof are Corollary 2.3, a *randomness shift* argument, taken from [20], which shifts randomness from the random perturbation to the seed, so to speak, and an absorbing structure.

Absorbers. We commence with a description of the absorbing structure, which can be viewed as a rainbow variant of the one used in [16] (see also [9]). Let H_1 and H_2 be edge-disjoint graphs on the same vertex-set and let $H = H_1 \cup H_2$. Let $\psi : E(H) \to \mathbb{N}$ be an edge-colouring. Let $P = p_1 p_2 \dots p_\ell$ be a path in H_1 which is rainbow under ψ and let $A = A(P) = \{\psi(p_i p_{i+1}) : 1 \leq i \leq \ell - 1\}$ be the set of colours seen along P under ψ. Let $I = I(P) = \{p_{3i} : 1 \leq i \leq \ell/3\} \setminus \{p_\ell\}$; note that $(\{w\} \cup N_P(w)) \cap (\{v\} \cup N_P(v)) = \emptyset$ for any two distinct vertices $w, v \in I$. For any two vertices $u, v \in V(H)$, set

$$B(u, v) = \{x \in N_{H_2}(u) \cap I : N_P(x) \subseteq N_{H_2}(v)\}.$$

We use sets of the form $B(u, v)$ in order to extend a given rainbow path by absorbing an external vertex so that the resulting extension remains rainbow. If for a vertex $v \in V(H) \setminus P$ there exists a vertex $p_j \in B(p_\ell, v)$ (note that p_ℓ is an end of P) such that $|\{\psi(p_{j-1}v), \psi(p_{j+1}v), \psi(p_\ell p_j)\} \setminus A| = 3$, then there is a rainbow path in H which is strictly longer than P. To see this, let p_j be as above. Then, $p_j \in N_{H_2}(p_\ell) \cap I$ and $p_{j-1}, p_{j+1} \in N_{H_2}(v)$. Moreover,

$$\{\psi(p_{j-1}v), \psi(p_{j+1}v), \psi(p_\ell p_j)\} \cap A = \emptyset$$

and

$$|\{\psi(p_{j-1}v), \psi(p_{j+1}v), \psi(p_\ell p_j)\}| = 3.$$

Therefore, the path $p_1 \dots p_{j-1} v p_{j+1} \dots p_\ell p_j$ forms a rainbow path in H with vertex-set $V(P) \cup \{v\}$. We say that p_j was *used to absorb* v.

Randomness shift. Next, we describe the *randomness shift* argument. Let r be a positive integer, let $R \sim \mathbb{G}(n, p)$, and let $\psi : E(R) \to [r]$ be an edge-colouring of R. Suppose that a.a.s. R contains a certain edge-coloured subgraph (in this paper it will be an almost spanning rainbow path, but we prefer to describe the argument in greater generality). Then we may assume that this subgraph (or some corresponding vertex-sets) is uniformly distributed over its copies in K_n (with an appropriate edge-colouring). Indeed, R and ψ can be generated as follows. First, generate a random graph $R' \sim \mathbb{G}(n, p)$ and colour its edges; denote the resulting colouring by ψ'. Next, permute the vertex-set of R' *randomly*; denote the resulting graph by R and the resulting edge-colouring by ψ. That is, choose a permutation $\pi \in S_n$ uniformly at random and set $R = ([n], \{\pi(u)\pi(v) : uv \in E(R')\})$ and $\psi(\pi(u)\pi(v)) = \psi'(uv)$ for every $uv \in E(R')$. The corresponding probability space coincides with $\mathbb{G}(n, p)$ with an appropriate edge-colouring; in particular $G' \subseteq R'$ is rainbow under ψ' if and only if $\pi(G') \subseteq R$ is rainbow under ψ. In this manner, the aforementioned edge-coloured subgraph of R is sampled uniformly at random through π.

We are now ready to prove our main result, namely, Theorem 1.1.

Proof of Theorem 1.1. Let d and α be as in the premise of the theorem and set $\varepsilon = d^3/200$. Let $K = K(\alpha, \varepsilon)$ be the constant whose existence is ensured by Corollary 2.3 and let $C = K + 1$. We expose $R \sim \mathbb{G}(n, C/n)$ in two rounds, that is, $R = R_1 \cup R_2$ where $R_1 \sim \mathbb{G}(n, K/n)$ and $R_2 \sim \mathbb{G}(n, p)$ for p which satisfies $1 - C/n = (1 - K/n)(1 - p)$; note that $p \geq 1/n$.

We first expose the edges of R_1 and colour them uniformly at random with colours from $[(1+\alpha)n]$; denote the resulting colouring by ψ. Set $\ell = (1 - \varepsilon)n$ (which we assume is an integer) and let $P = p_1 p_2 \ldots p_\ell$ be a path in R_1 which is rainbow under ψ; such a path exists (a.a.s. in R_1) by Corollary 2.3.

Next, we use the edges of $H \in \mathcal{G}_{d,n}$ in order to extend P to a rainbow path on $n-2$ vertices. Let $I = \{p_{3i} : 1 \leq i \leq \ell/3\} \setminus \{p_\ell\}$. We begin by proving that, with respect to H, P and I, the set $B(u, v)$ is large for every $u, v \in V(H)$ (i.e., here R_1 assumes the role of H_1 in the definition of $B(u, v)$ and H assumes the role of H_2); this is done without revealing the colours of the edges of H.

Claim 3.1. *Asymptotically almost surely $|B(u, v)| \geq d^3 n/100$ holds for every $u, v \in V(H)$.*

Proof. Fix some $u, v \in V(H)$. As explained above, we may assume that a random permutation $\pi : V(R_1) \to V(H)$ maps P to a path P'. We assume that the images $\pi(p_1), \pi(p_2), \ldots, \pi(p_\ell)$ are determined (randomly) first in this order, and then the images $\pi(w)$ are set for every $w \in V(R_1) \setminus V(P)$ in an arbitrary order. For every $1 \leq i \leq \ell$, let X_i denote the indicator random variable for the event $\pi(p_i) \in N_H(u)$ and let Y_i denote the indicator random variable for the event $\pi(p_i) \in N_H(v)$. For every i such that $p_i \in I$, let $Z_i = Y_{i-1} X_i Y_{i+1}$. Then Z_i is the indicator random variable for the event $\pi(p_i) \in B(u, v)$ and thus $|B(u, v)| = \sum Z_i$, where the sum is extended over all $1 \leq i \leq \ell$ for which $p_i \in I$. Let A_u (respectively A_v) be the event that $|N_H(u) \cap \{\pi(p_1), \ldots, \pi(p_{n/3})\}| \geq |N_H(u)|/2$ (respectively $|N_H(v) \cap \{\pi(p_1), \ldots, \pi(p_{n/3})\}| \geq |N_H(v)|/2$). Note that $|N_H(u) \cap \{\pi(p_1), \ldots, \pi(p_{n/3})\}|$ (and its counterpart for v) is distributed hypergeometrically owing to the randomness shift argument. A straightforward application of Chernoff's bound for the hypergeometric distribution then shows that $\mathbb{P}(A_u \cup A_v) = o(1)$. Hence, for the remainder of the proof we will assume that $A_u^c \cap A_v^c$ holds.

Recall that $(\{w\} \cup N_P(w)) \cap (\{v\} \cup N_P(v)) = \emptyset$ holds for any two distinct vertices $w, v \in I$. Hence

$$\mathbb{P}(Z_i = 1) \geq \frac{|N_H(v)| - \sum_{j=1}^{i-2} Y_j}{n} \cdot \frac{|N_H(u)| - \sum_{j=1}^{i-1} X_j}{n} \cdot \frac{|N_H(v)| - \sum_{j=1}^{i} Y_j}{n} \geq \frac{d^3}{8}.$$

holds for every $1 \leq i \leq n/3 - 1$ for which $p_i \in I$, regardless of the value of Z_j for any $j \neq i$ for which $p_j \in I$. Therefore

$$\mathbb{P}\left(|B(u, v)| < d^3 n/100\right) \leq \mathbb{P}\left(\mathrm{Bin}\left(n/10, d^3/8\right) < d^3 n/100\right) < e^{-\Omega\left(d^3 n\right)},$$

where the last inequality holds by a standard application of Chernoff's bound. Finally, a union bound over all pairs $u, v \in V(H)$ shows that the probability that there exists such a pair for which $|B(u, v)| < d^3 n/100$ is $o(1)$. □

Let $P_0 = P$ (formally, $P_0 = \pi(P)$, but we avoid using this more accurate notation for the sake of clarity and simplicity of the presentation) and let $x, y, v_1, \ldots, v_s$ be the vertices of $V(H) \setminus V(P_0)$. We extend P_0 (via the edges of the seed H) by absorbing $v_1, \ldots, v_s$ one by one whilst keeping p_1 as one of the ends throughout. Assume that for some $i \geq 0$ the path P_i with vertex-set $V(P_0) \cup \{v_1, \ldots, v_i\}$ has already been built and consider the subsequent extension of P_i into P_{i+1}, obtained by absorbing v_{i+1}. For every $1 \leq j \leq i$, let u_j be the vertex of I that was used to absorb v_j. Let z denote the endpoint of P_i that is not p_1 (note that $z = p_\ell$ if $i = 0$ and $z = u_i$ otherwise).

Let $B_i(z, v_{i+1}) = B(z, v_{i+1}) \setminus \{u_1, \ldots, u_i\}$ (in particular, $B_0(z, v_1) = B(z, v_1)$), and note that

$$|B_i(z, v_{i+1})| \geq |B(z, v_{i+1})| - i \geq d^3 n/100 - \varepsilon n \geq d^3 n/200 \tag{1}$$

holds a.a.s. for any $0 \leq i \leq s$, by Claim 3.1 and the choice of ε. This removal of vertices that were previously used for absorption is crucial in two respects. First, absorbing triples cannot be reused. Second, and this unfolds more explicitly towards the end of the proof, there is a need to keep track over edges for which the random colouring ψ has already been exposed and where randomness still lies, so to speak.

For every vertex $p_j \in B_i(z, v_{i+1})$, we now expose the colours of the edges $p_j z$, $p_{j-1} v_{i+1}$, and $p_{j+1} v_{i+1}$ and extend the colouring ψ to these edges; note that, crucially, the colours of $E_H(v_{i+1}, P_i) \cup E_H(z, I)$ were not previously exposed. An exception to this rule occurs if one of these edges is in R_1 as well. However, a standard calculation shows that a.a.s. the maximum degree of R_1 is $o(\ln n)$, implying that a.a.s. $e_{R_1}(v_{i+1}, P_i) + e_{R_1}(z, I) \leq \ln n$. If there exists a vertex $p_j \in B_i(z, v_{i+1})$ such that $\{\psi(p_{j-1} v_{i+1}), \psi(p_{j+1} v_{i+1}), \psi(z p_j)\} \cap \{\psi(e) : e \in E(P_i)\} = \emptyset$ and $|\{\psi(p_{j-1} v_{i+1}), \psi(p_{j+1} v_{i+1}), \psi(z p_j)\}| = 3$, then it can be used to absorb v_{i+1} as explained above. Using once again the fact that $(\{w\} \cup N_P(w)) \cap (\{v\} \cup N_P(v)) = \emptyset$ holds for any two distinct vertices $w, v \in I$, it follows that the probability that no such vertex exists is at most

$$\left(1 - \left(\frac{\alpha}{1+\alpha}\right)^3\right)^{|B_i(z, v_{i+1})| - \ln n} \leq e^{-\alpha^3 (1+\alpha)^{-3} d^3 n/201} = o(1/n).$$

Since this holds for every $0 \leq i \leq s$, a union bound shows that the probability that we fail to absorb at least one of the vertices $v_1, \ldots, v_s$ is o(1).

Denote the ends of the resulting rainbow path P_s by p_1 and p_{n-2}. We now use the edges of R_2 in order to extend P_s into a rainbow Hamilton cycle. Let $X = B_s(p_1, x)$ and let $Y = B_s(p_{n-2}, y)$. The sizes of these sets are captured by (1). Note that the colours of $E_H(\{x, y\}, V(H))$ were not yet exposed. Similarly, the colours of $E_H(p_1, X) \cup E_H(p_{n-2}, Y)$ were not yet exposed. Indeed, neither X nor Y meet the set $\{u_1, \ldots, u_s\}$, as by definition of the B_i-sets, members of $\{u_1, \ldots, u_s\}$ are repeatedly removed. Moreover, as the B_i-sets start from sets that do not meet the set $\{v_1, \ldots, v_s\}$ and in subsequent absorption steps are only refined, neither X nor Y meet the set $\{v_1, \ldots, v_s\}$. The claim follows by observing that throughout the absorption process the sole edges of H whose colours are exposed are incident with $\{p_\ell, u_1, \ldots, u_s\} \cup \{v_1, \ldots, v_s\}$. Other relevant edges whose colours were already exposed, are the edges of $E_{R_1}(X, Y)$ and the edges of $E_{R_1}(x, P_s) \cup E_{R_1}(y, P_s) \cup E_{R_1}(p_1, X) \cup E_{R_1}(p_{n-2}, Y)$. A standard application of Chernoff's bound and a union bound over all pairs of sets of appropriate sizes shows that a.a.s. $e_{R_1}(X, Y) \leq 2K|X||Y|/n = o(|X||Y|)$. Moreover, a standard calculation shows that a.a.s. $e_{R_1}(x, P_s) + e_{R_1}(y, P_s) + e_{R_1}(p_1, X) + e_{R_1}(p_{n-2}, Y) \leq \ln n$.

Let $X' := \{p_i \in X : \{p_i p_1, p_{i-1}x, p_{i+1}x\} \cap E(R_1) = \emptyset\}$ and $Y' := \{p_i \in Y : \{p_i p_{n-2}, p_{i-1}y, p_{i+1}y\} \cap E(R_1) = \emptyset\}$; as noted in the preceding paragraph $|X'| \geq |X| - \ln n$ and $|Y'| \geq |Y| - \ln n$ hold asymptotically almost surely. Expose the edges of R_2 with one endpoint in X' and the other in Y' (that are not edges of R_1) and extend the colouring ψ to these edges, and to the edges in $E_H(x, P_s) \cup E_H(y, P_s) \cup E_H(p_1, X') \cup E_H(p_{n-2}, Y')$ (that are not edges of R_1). If there exists an edge $p_i p_j \in E_{R_2}(X', Y') \setminus E(R_1)$ such that

$$\{\psi(p_i p_j), \psi(p_1 p_i), \psi(x p_{i-1}), \psi(x p_{i+1}), \psi(p_{n-2} p_j), \psi(y p_{j-1}), \psi(y p_{j+1})\} \cap \{\psi(p_t p_{t+1}) : 1 \leq t \leq n-3\} = \emptyset$$

and

$$|\{\psi(p_i p_j), \psi(p_1 p_i), \psi(x p_{i-1}), \psi(x p_{i+1}), \psi(p_{n-2} p_j), \psi(y p_{j-1}), \psi(y p_{j+1})\}| = 7,$$

then (assuming without loss of generality that $i < j$) the sequence

$$p_i p_1 \ldots p_{i-1} x p_{i+1} \ldots p_{j-1} y p_{j+1} \ldots p_{n-2} p_j p_i$$

forms a Hamilton cycle of $H \cup R$ which is rainbow under ψ.

Since P_s is coloured using $n-3$ colours, the colours for the aforementioned seven edges (assuming $p_i p_j$ exists) can be chosen from a set of at least αn colours. The probability that such an edge $p_i p_j$ does not exist in R_2 or that it does exist in R_2 and a colour clash occurs along the aforementioned seven edges as detailed above, is at most

$$\left(1 - \frac{O_\alpha(1)}{n}\right)^{|X'||Y'| - (|X' \cap Y'| + 1)^2/2 - e_{R_1}(X', Y')} \leq \left(1 - \frac{O_\alpha(1)}{n}\right)^{|X'||Y'|/3} \leq e^{-\Omega_{\alpha,d}(n)} = o(1),$$

where the second inequality holds by (1). □

Acknowledgement

We would like to thank the anonymous referees for helpful comments.

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